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ON THE PROBLEM OF THE CONVERGENCE OF THE EIKONAL EXPANSION FOR SYNTHETIC SEISMOGRAMS

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ABSTRACT

In this paper, we discuss the convergence of the eikonal method, when applied to computing synthetic seismograms, by means of a comparison with the exact solution, computed through the Cagniard method. We prove that the eikonal method may yield a divergent series, for sufficiently large $t-\tau$.

INTRODUCTION AND STATEMENT OF THE PROBLEM

The generalized ray theory in a broader sense, which is known as the eikonal method in French literature, is a most powerful method for computing synthetic seismograms (Babich and Alekseev, 1958; Hron, 1968).

It is based upon the following set of fundamental hypotheses:

1. The seismic displacement is expressed as an expansion

$$\mathbf{U}(x,y,z;t) = \sum_{k=0}^{\infty} f_k(t-\tau(x,y,z)) \,.\, \mathbf{W}_k(x,y,z). \tag{1}$$

2. Functions $\mathbf{W}_k$ and τ are independent of time t and of the excitation function f_o as well, provided the symmetry of the seismic source is kept.

3. Functions f_k are subsequent primitives of the excitation function f_o

$$f_k' = f_{k-1}.$$

On the basis of these hypotheses, which enable one to separate propagation $(\mathbf{W},\tau)$ and excitation (f_k), one may derive a computation of the $\mathbf{W}_k$'s: Inside a continuous medium, one solves partial derivative equations, and at a surface of discontinuity, one may use the Zöppritz coefficients, of which a general algebraic expression may be derived (Hron, 1968; Okal, 1972a).

However, the various $\mathbf{W}_k$'s in expression (1) are yielded in a *recurrent* way; one has a hard time computing $\mathbf{W}_2$ or $\mathbf{W}_3$ and it is impossible to give a formal algebraic expression for $\mathbf{W}_k$.

Therefore, the convergence of the series over k in (1) is difficult to study directly. Babich (1961) proved that there always is some instant $t-\tau$ small enough to secure the convergence. Pod"yapol'skii (1966) gave a general expression for a time $t-\tau$ when it still converges. In the present paper, we shall prove that, under certain circumstances, there always exists a time $t-\tau$ large enough when the series diverges. In order to be able to discuss this convergence, we will take a very particular situation, when the exact solution for $U(x,y,z;t)$ can be computed by the Cagniard method (Cagniard, 1939).

COMPUTATION BY THE CAGNIARD METHOD

Let us assume (see Figure 1) a semi-infinite medium, under a vacuum. Let us take a spherical source at depth h with an excitation function f_o defined as

$$f_o(\xi) = 1 \quad \text{if} \quad 0<\xi<T;$$

$$\text{otherwise} \quad f_o(\xi) = 0.$$

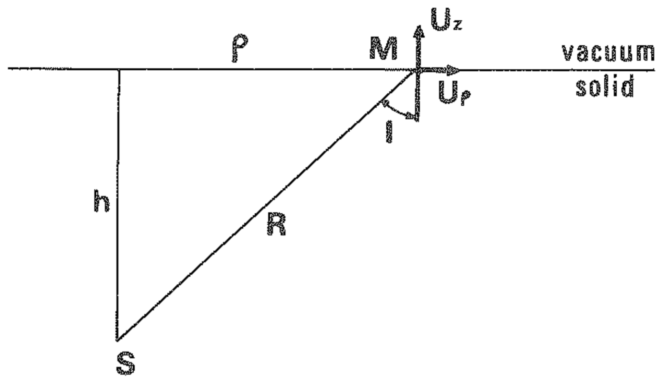

FIG. 1. Particular situation assumed to compare the two methods.

This particular choice of the excitation function leads to the following theorem:

The expansion (1) is, if it exists, unique, and is obtained by expanding Cagniard's exact solution into powers of the variable $t-\tau(x,y,z)$.

Let us now pay attention to the horizontal displacement U_ρ at point M located on the interface. Cagniard's method yields

$$U_\rho = A\cdot f_o(t-R/\alpha)+B\cdot \int_{R/\alpha}^{t} S_\rho(v) f_o(t-v)\, dv.$$

Here, A and B are two constants with respect to t and the "propagation factor" $S_\rho(v)$ for a propagation time v is

$$S_\rho(v) = \int \frac{b \quad u^3 \quad du}{D(u)\cdot\{u^2\rho^2+(v-ah)^2\}^{3/2}}, \tag{2}$$

where

$$a = (u^2+1/\alpha^2)^{1/2}; \qquad b = (u^2+1/\beta^2)^{1/2};$$

$$D(u) = [u^2+1/(2\beta^2)]^2-abu^2,$$

α and β being the P-wave and S-wave velocities.

The integral in (2) is taken along a path of integration which is described in Figure 2 and which is composed of:

1. A great loop of large radius (R),
2. Two small loops around the poles of D: (P,P'), and
3. A contour surrounding the cut provided from $-i/\beta$ to $+i/\beta$ in order to take into account the various determinations of the square roots in a and b. Let this contour be E.

Discussion of the Convergence

The first term in U_ρ is a zeroth order term of the series and it can have no influence on its convergence. Then, the remaining integral may be rewritten, by setting $\xi = t-R/\alpha$

$$U_\rho'(x,y,z,\xi) = \int_0^\xi f_o(\xi-\eta) S_\rho(\eta+R/\alpha)\cdot d\eta$$

$$= \int_0^\xi S_\rho(\eta+R/\alpha)\cdot d\eta.$$

The convergence of U_ρ', expanded on powers of ξ, will be identical to that of S_ρ, considered as a function of η.

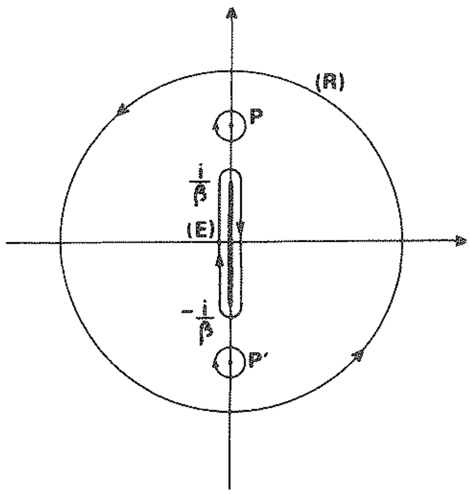

FIG. 2. Path of integration in the plane of the complex variable u.

This enables us to study only the three integrals: $S_\rho{}^R$, $S_\rho^{PP'}$, and $S_\rho{}^E$.

1. $$S_\rho{}^R = -\frac{2 \sin I\, v^2}{R^2 \cdot (v^2-1)}$$

is a constant, which cannot change the convergence of S_ρ.

2. Then, the Rayleigh contribution may be written as

$$S_\rho^{PP'} = C \cdot Im\{[r^2-\mu^2+\cos^2 I+2\, \mathrm{i}r \cos I.(\mu^2-1)^{1/2}]^{-3/2}\}.$$

Here, $r = \alpha{\cdot}v/R$ is a dimensionless variable and C is a constant with respect to time. $v = \alpha/\beta$ is the ratio of the P- and S-wave velocities. The quantity between brackets is expandable into series of powers of $x = r-1$, its radius of convergence being $\Psi = \mu - \sin I$, where μ is the ratio of the P-wave and Rayleigh-wave velocities: $\mu = 1.883\ldots.$ It may easily be proved (Okal, 1972a) that $S_\rho^{PP'}$ has the same radius of convergence.

3. Then let us pay attention to the last term: $S_\rho{}^E$. A change of variable in the integral (2) leads to the following expression for $S_\rho{}^E$ (Cagniard, 1939; Okal, 1972a)

$$S\rho^E = -\frac{\rho i}{\pi}$$

$$\int_{-(v^2-1)^{\frac{1}{2}}}^{(v^2-1)^{\frac{1}{2}}} \frac{\lambda\, d\lambda(1+\lambda^2)\cdot(1+\lambda^2-v^2/2)^2(v^2-1-\lambda^2)^{1/2}}{\{(1+\lambda^2-v^2/2)^4+\lambda^2(\lambda^2+1)^2(v^2-1-\lambda^2)\}\cdot\{(\alpha v+i\lambda h)^2-(\lambda^2+1)\rho^2\}^{3/2}}.$$

This quantity may be proved to be expandable into a power series of $x = r-1$. This comes from the identity

$$(\alpha v+i\lambda h)^2-(\lambda^2+1)\rho^2 = R^2(x+s_1)\cdot(x+s_2),$$

where the roots s_1, s_2 are obtained as

$$s_1, s_2 = -1-(i\lambda h)/R \pm (\lambda^2+1)^{1/2}\cdot \rho/R.$$

If m and M denote the minimum and maximum values of the modulus of s_1 and s_2 when λ varies on the integration segment, then one proves (Okal, 1972a) that

$$\{(x+s_1)\cdot(x+s_2)\}^{-3/2} = \Sigma c_n \cdot x^n,$$

with

$$c_n < \frac{2n+1}{m^3}\cdot(2M/m^2)^n,$$

and that $m \neq 0$.

Therefore, as the convergence is proved to be uniform with respect to λ, $S_p{}^E$ is expandable into a power series of x with a radius of convergence not smaller than $m^2/2M$. Then, as D does not go to zero while λ varies on the integration segment, it is impossible for $S_p{}^E$ to have as its own pole the pole of $S_p^{PP'}$ of smallest modulus, which was responsible for the radius of convergence of $S_p^{PP'}$ (Okal, 1972a).

Accordingly, the radius of convergence of the sum

$$S_p = S_p{}^R + S_p^{PP'} + S_p{}^E$$

cannot be greater than that of $S_p^{PP'}$, namely $\Psi = \mu - \sin I$.

Therefore, if $t > t_1$, with

$$t_1 = \frac{R}{\alpha}\cdot(1+\mu-\sin I),$$

the series (1) diverges.

Generalization

These results may be generalized as follows

1. If one considers a time $t > T$ and if $T < t_1$, then the expansion

$$\Sigma\, W_k \cdot \frac{(t-\tau)^k}{k\,!}$$

is to be replaced by

$$\Sigma\, W_k \cdot \frac{(t-\tau)^{k-1}}{(k-1)\,!},$$

which obviously has the same convergence.

2. If one deals with any function f_o which remains larger than a rectangular function g_o (see Figure 3), general theorems on series prove that, outside the radius of the convergence of the series built on the g's, that built on the f's diverges as well.

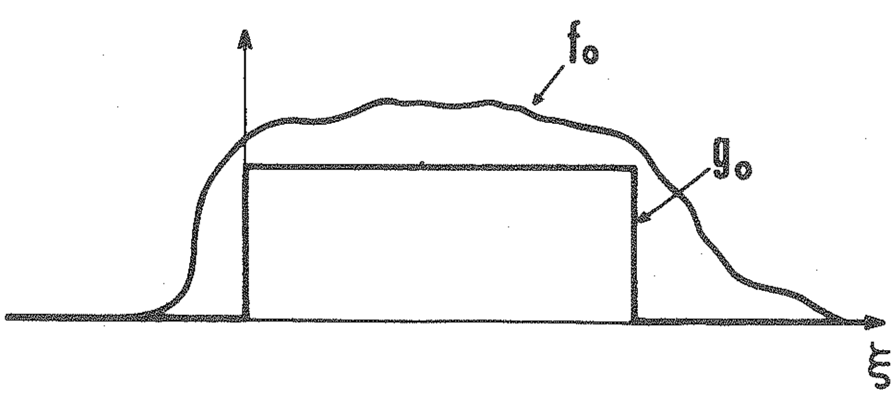

FIG. 3. Example of functions f_o and g_o.

3. The study of the vertical component U_z of the seismic displacement leads to the same radius of convergence as that of U_ρ.

DISCUSSION

The results obtained in this paper are very similar to those derived by Pod"yapol'skii (1966). The study by Pod"yapol'skii stated that, inside a certain region in the $t-\tau$ space, the expansion is convergent. He pointed out that, outside this region, the series may still be convergent, owing to special geometry of the problem, leading to annihilation of singularities.

Although the material in the present paper is certainly derivable from Pod"yapol'skii's results, we obtain it from Cagniard's method by a very simple way. Furthermore, we prove that in the chosen particular case, and under very little stringent hypothesis on the excitation function, the series (1) is actually divergent when $t>t_1$. Therefore, the eikonal method appears as a powerful method to investigate wave-front intensities; it should not be used to compute a seismogram in its further steps, and especially its coda.

Direct comparisons of computer-achieved calculations by both this method and Cagniard's agree with this last result.

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