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# A PHYSICAL CLASSIFICATION OF THE EARTH'S SPHEROIDAL MODES

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This paper shows that five different families of spheroidal modes can be isolated, namely: 1) Inner Core and Stoneley modes ("$K$" modes); 2) "$V$" (vertical) modes, with mainly vertical displacement; 3) "$C$" (Colatitudinal) modes, with mainly horizontal displacement; 4) "$R$" (Rayleigh) modes, in which the horizontal and vertical displacements are totally coupled, and 5) "$H$" (Hybrid) modes, with intermediate coupling. $V$ and $C$ modes occur at high phase velocities, $R$ modes at low phase velocities, and $H$ modes at intermediate ones. Each of the families of modes has distinctly different properties, including group velocity, $Q$, and excitation functions.

Except for $H$ modes, these families are arranged in "pseudo-overtone" branches, along which physical properties vary smoothly. A theoretical description of the properties of $V$, $C$ and $K$ modes is given, using the simplified model of a homogeneous, non-gravitating Earth. Two important observations are explained, using this model: i) The solution for $C$ modes at low values of $l$ are identical to the ones for corresponding $T$ (Torsional) modes, and have, therefore, the same eigenperiods are relative excitation functions, and ii) the radial modes ${}_nS_0$ are the $l=0$ members of the $V$ family, and their apparent scarcity results simply because only that family has modes with $l=0$. Furthermore, the group velocity of $K$, $C$, $V$ and $R$ modes is shown to be consistent with the physical concept of dispersion along a pseudo-overtone branch. An interpretation of the existence of the different families in terms of an increase in mode-coupling with angular order is presented.

A formal classification of the spheroidal modes into 5 families is made, and a new nomenclature is proposed, which is closely related to their physical properties.

## *1. Introduction*

The purpose of this paper is to disentangle the physical properties of the spheroidal modes and to propose a classification and possible new nomenclature for them. In the conventional nomenclature, modes of similar an-

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gular order number $l$ are sorted by increasing frequencies. The physical properties of the modes (group velocity $U$, attenuation factor $Q$, particle motion at the surface, excitation functions) can vary dramatically with small changes in either $l$ or the overtone number $n$. It is shown in this paper that most of the spheroidal modes can be classified into several families offering regular, although different, trends in their physical properties. Specifically, one can isolate the Inner Core and Stoneley modes; then, among the remaining modes, for low $l$, or equivalently, at high phase velocities, there are two completely different sets of spheroidal modes. The first is a family of highly attenuated modes, for which the group velocity is slow (usually $\leq 5$ km/s), and the main component of the displacement colatitudinal. The eigenfunctions of these modes (and therefore, their periods, group velocities, $Q$'s and excitation functions) are strikingly similar to those of torsional modes of the same angular order. The second family consists of modes with higher $Q$'s, whose group velocities are higher (usually $\geq 10$ km/s), for which the displacement is mainly vertical, and whose physical properties are continuous with those of the so-called "radial" modes ${}_nS_0$. At high $l$ (lower phase velocities), total coupling occurs between the vertical and colatitudinal modes, leading to a single family, whose physical properties are extremely regular, and can directly be compared to Rayleigh waves. At intermediate $l$, coupling occurs irregularly and it is not possible to define any strong trend in the physical properties.

The first section of this paper describes the irregular variations in the modes' properties with small variations of either $n$ or $l$, when the conventional nomenclature is used. The empirical analysis of a set of computed data introduces the idea of *several* families of spheroidal modes. In the second section, we adapt the results of ALTERMAN *et al.* (1959) to the simplified case of a homogeneous Earth and of total decoupling between radial and horizontal displacements. We extend the results of ANDERSSEN *et al.* (1975) and of GILBERT (1975), to discuss the values of the group velocities, inside the various families at low $l$. A comparison is made with the values computed for a realistic Earth model. The third section formally presents the classification of spheroidal modes and proposed new nomenclature.

## 2. *A Critical Review of Spheroidal Modes*

The problem of the vibrations of an elastic sphere dates back to LAMB (1882). A complete review of the literature on this subject is beyond the scope of this paper and we shall only summarize the following milestones in the development of mode theory: LOVE (1911), and later PEKERIS and JAROSCH (1958) discussed the eigenfunctions of a uniform, gravitating sphere; ALTERMAN *et al.* (1959) first calculated the excitation coefficients of the various spheroidal oscillations of the Earth for a simple source; SATÔ and USAMI (1962a, b, c) and LANDISMAN *et al.* (1970) extensively studied the problem of the oscillations of a homogeneous sphere. Ray-mode duality was also investigated by BRUNE (1964), BEN-MENAHEM (1964), and more recently WOODHOUSE (1977).

SAITO (1967) presented general results, applicable to any seismic source, and introduced a variational method of solving the differential equations. KANAMORI and CIPAR (1974) gave a simplified expression of the excitation of both $T$ and $S$ modes by any double-couple, and KANAMORI and STEWART (1976) introduced asymptotic expansions, which help to avoid having to sum a forbidding number of modes at higher frequencies. The extension of their method to overtones is necessary to apply their formalism to problems involving deep source and/or body waves (OKAL and GELLER, 1978a, b). Experimental identifications of the normal modes of the Earth were systematically carried out by DZIEWONSKI and GILBERT (1972) and GILBERT and DZIEWONSKI (1975). A theoretical investigation of the asymptotic behavior of ${}_nS_l$, at constant $l$, was given by ANDERSSEN *et al.* (1975) and by GILBERT (1975). However, these authors have limited their investigation to $\omega\to\infty$ at constant $l$, thus neglecting the study of the modes' group velocity, which is of crucial importance in the approach of Kanamori and Stewart, as the group velocity is the quantity associated with the variation of physical properties with wavenumber (or equivalently angular order).

### 2.1 *The following section briefly summarizes SAITO's (1967) results in the simplified version of KANAMORI and CIPAR (1974)*

The displacement $\boldsymbol{u}$ at a point $(r, \theta, \phi)$ in the Earth, generated by an earthquake, can be expanded into a sum of the normal modes ${}_nS_l^m$ of the Earth ($l$: angular order; $m$: azimuthal number; $n$: overtone number); the amplitude of excitation of a given mode by a particular source can be separated into radiation pattern factors $(p_R, q_R, s_R)$, depending only on the mechanism of the earthquake, and excitation coefficients $(K_0, K_1, K_2; N_0)$, depending only on the source depth and on the particular mode considered. The notation in this paper will always be that of SAITO (1967) and KANAMORI and CIPAR (1974). However, the angular order number will always be $l$. Also, $N_0$ is the excitation coefficient for a purely compressional source, adapted from TAKEUCHI and SAITO (1972):

$$N_0 = -\frac{2l+1}{4\pi\,{}_n\omega_l^2(I_1+L^2I_2)}\cdot D(r_s)\,, \tag{1}$$

where $D(r_s)Y_l^m(\theta, \phi)=\varepsilon_{ii}=\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}$ is the trace of the strain tensor or deformation:

$$D(r_s)=\frac{4\mu_s}{(\lambda_s+2\mu_s)r_s}y_1(r_s)+\frac{y_2(r_s)}{\lambda_s+2\mu_s}-\frac{L^2}{r_s}\cdot\frac{2\mu_s}{\lambda_s+2\mu_s}y_3(r_s)\,. \tag{2}$$

Following KANAMORI and CIPAR (1974) and KANAMORI and STEWART (1976), we will always normalize $y_1(a)$ to unity and the excitation functions will always be computed assuming a double-couple moment of $10^{27}$ dynes-cm (or a purely compressional moment of $10^{27}$ dynes-cm for each of the three equivalent dipoles in the case of a compressional source).

The computed data set used in this study consists of some 5,200 theoretical eigenfunctions for the Earth, computed by BULAND and GILBERT (1976) for model 1066A (GILBERT and DZIEWONSKI, 1975). This data set includes 1,936 torsional and 3,271 spheroidal modes, representing all solutions with angular order less than 151 and periods greater than 45 seconds. For periods larger than 150 seconds, the solutions for all torsional modes and most spheroidal ones were checked against an independent recomputation by the author, using model C2 (ANDERSON and HART, 1976), and the program developed by KANAMORI and ABE (1968). The eigenfunctions were processed to obtain the periods ($T$), phase velocities ($C$), group velocities ($U$), attenuation factors ($Q$), surface transverse displacements ($y_3(a)$, with $y_1(a)=1$, to be abbreviated below as $y_3$). The excitation functions $N_0$, $K_0$, $K_1$ and $K_2$ ($L_1$ and $L_2$ in the case of $T$ modes) were obtained for 23 standard focal depths between 0 and 750 km. The values of these parameters represent more than 400,000 numbers and cannnot be listed here. We present only typical examples.

It should be noted that the group velocity $U$ is computed here as outlined by JEFFREYS (1961):

$$U=(\mathscr{J}_1+\mathscr{J}_2/k)/\mathscr{J}_0 C\,, \tag{3}$$

where $k=(l+1/2)/a$ is the wavenumber, and the $\mathscr{J}$'s are energy integrals involving the eigenfunction and its derivative with respect to $r$. The concept of group velocity assumes the existence of a set of modes (generating a wave), whose physical properties vary smoothly enough that they can be easily followed and considered as continuous with frequency, despite the discrete layout of the modes. The group velocity $d\omega/dk$ is then used in the interpolation of physical properties of the modes (KANAMORI and STEWART, 1976). In the case of spheroidal modes, this assumption might sometimes be inappropriate along an overtone branch $n=$constant. We shall return to this point below.

The values of $Q$ were obtained both from the MM8 model of ANDERSON *et al.* (1965), and from the more recent SL2 model described by ANDERSON and HART (1978). The primary difference between these two models is the presence of a zone of low $Q$ at the base of the mantle, and the low values of $Q$ in the inner core in model SL2.

*2.2 Difficulties with the conventional nomenclature ${}_nS_l$*

In the conventional nomenclature, modes of identical $l$ are assigned an overtone number by increasing frequency: the mode with the longest period is called ${}_0S_l$, the next one ${}_1S_l$, and so on. (The first mode with $l=1$, ${}_0S_1$, which represents a rigid body translation of the whole Earth, and for which ${}_0\omega_1=0$, is not usually included in any compilation of $S$ modes, although it is tacitly part of the conventional nomenclature.) A similar method is used for torsional modes ${}_nT_l$. However, the torsional nomenclature does not usually include the inner core torsional oscillations, for which no displacement can be transmitted through the liquid outer core, and which therefore, can be neither excited in the mantle nor observed at the surface of the Earth. In the case of spheroidal modes the following problems are encountered:

i) It is no longer true that the vertical eigenfunction, $y_1(r)$, of ${}_nS_l$ has $n$ zero-crossings along the radius of the Earth. This property, which holds for $T$ modes, remains true in the case of $S$ modes only for $n=0$, for $l=0$, and, locally, for other values of $n$ and $l$.

ii) For $l=0$, the number of ${}_nS_0$ modes over a given range of frequencies is much smaller (roughly 2.5 times) than the corresponding number of their neighbors ${}_nS_1$ or ${}_nS_2$: For example, ${}_{20}S_0$ has a period of 57.7 s, which is comparable to the period of ${}_{60}S_1$: 56.4 s. Thus, one cannot link those so-called "radial" modes, ${}_nS_0$, with the other spheroidal modes, resulting in their being

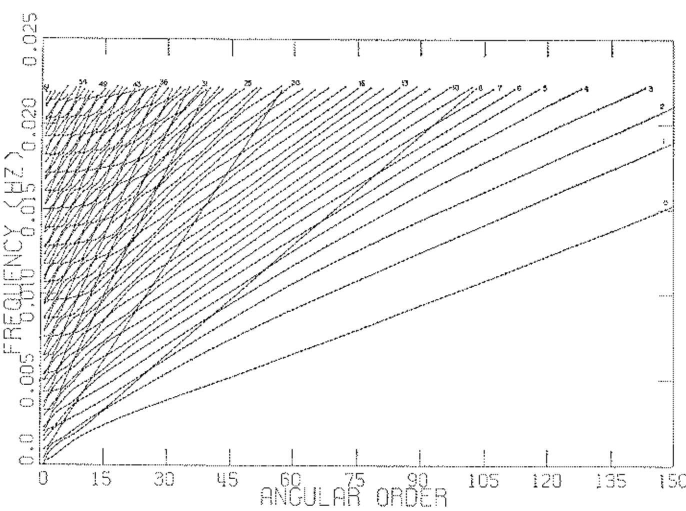


Fig. 1. Frequency vs. angular order plot of the spheroidal modes used in this study, as computed from model 1066A. Overtones are traced and labeled as resulting from the conventional nomenclature; radial modes ($l=0$) are not integrated in overtone branches.

isolated (e.g. PEKERIS and JAROSCH, 1958), and often listed separately in different tables (e.g. ANDERSON and HART, 1976, 1978).

iii) GILBERT and DZIEWONSKI (1975) have pointed out that the conventional nomenclature for certain modes may depend upon the Earth model used to compute their periods: For example, ${}_{26}S_1$ and ${}_{27}S_1$ are interchanged if one uses 1066B instead of 1066A (ANDERSSEN *et al.*, 1975).

iv) However, the most important drawback of the conventional nomenclature is the absence of continuity in the physical properties of the modes along overtone branches: Figure 1 is a plot of the eigenfrequencies of the spheroidal modes (computed from model 1066A) used in this study, the over-

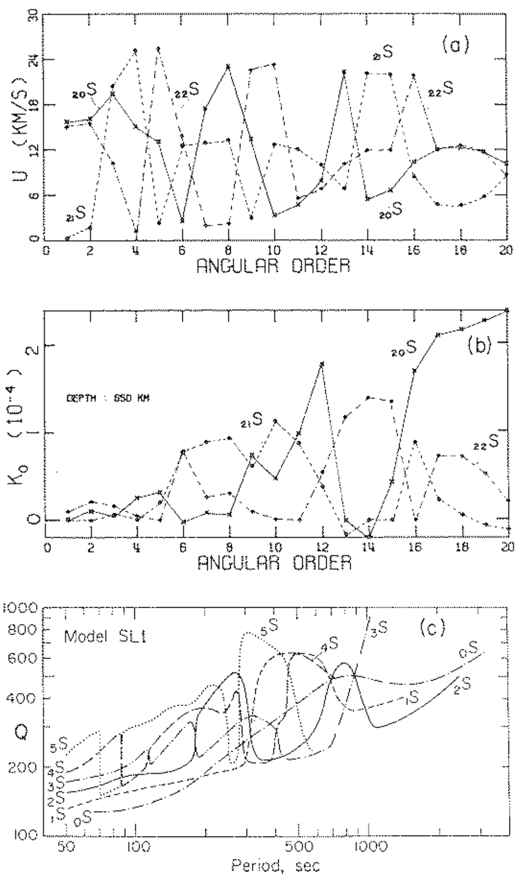


Fig. 2. A plot of several physical properties of spheroidal modes along conventional overtone branches: (a) Group velocity $U$ along ${}_{20}S_l$, ${}_{21}S_l$ and ${}_{22}S_l$ ($l$=1–20); (b) Excitation coefficient $K_0$ at a depth of 650 km, along ${}_{20}S_l$, ${}_{21}S_l$ and ${}_{22}S_l$; (c) $Q$ along the lower overtones ${}_0S_l$ to ${}_5S_l$ as a function of period. (This last plot is taken from ANDERSON and HART (1978).)

Table 1. An example of discrepancies between dispersion group velocities $U^*$ and energy integral group velocities $U$.

| Mode | Periods (s) | $U$ (km/s) | $Q_{SL2}$ | $y_3$ |
|---|---|---|---|---|
| ${}_{50}S_2$ | 55.19 | 16.43 | 984 | 0.076 |
| ${}_{50}S_3$ | 54.96 | 0.38 | 258 | −7.98 |
| ${}_{51}S_2$ | 54.98 | 0.43 | 260 | −4.48 |
| ${}_{51}S_3$ | 53.95 | 15.63 | 980 | 0.003 |

Dispersion group velocities $U^*$:
${}_{50}S_2$–${}_{50}S_3$: 3.04 km/s ${}_{51}S_2$–${}_{51}S_3$: 13.90 km/s
${}_{50}S_3$–${}_{51}S_2$: 0.28 km/s ${}_{50}S_2$–${}_{51}S_3$: 16.67 km/s

tones being traced and labeled as they result from the conventional nomenclature: They clearly display a very rugged behavior of the periods $T$ along overtone branches. Similarly, Fig. 2 shows the variation of the group velocity $U$, and of $K_0$ (for a depth of 650 km), along the three branches ${}_{20}S_l$, ${}_{21}S_l$ and ${}_{22}S_l$, and of $Q$ along the lower overtone branches ${}_0S_l$ to ${}_5S_l$. These curves are totally irregular, and no physical interpretation of them is apparent. Such irregular behavior is not observed for torsional modes, which exhibit smoothly varying properties at similar periods and/or angular numbers.

v) Also, in some cases, the group velocity $U$ computed from JEFFREYS' (1961) formula does not represent a correct approximation to the dispersion $d\omega/dk$ along an overtone branch $n$=constant. As an example, in Table 1, we list the properties of four adjacent modes (${}_{50}S_2$, ${}_{50}S_3$, ${}_{51}S_2$, ${}_{51}S_3$) and compare the theoretical group velocities $U$ (computed from Jeffreys' integrals) with "physical" group velocities $U^*$, obtained by approximating the definition of group velocity:

$$U^* = d\omega/dk\ , \tag{4}$$

by $U^* = a\ ({}_n\omega_{l+1} - {}_n\omega_l)$ along what is commonly called an overtone branch, that is a set of normal modes with constant $n$. The agreement is seen to be very poor. Also, the other physical properties (such as $Q$, $y_3(a)$, $K_0$, ...) strongly vary along the overtone branches ($n=50$ or $n=51$). Again, this behavior is absent from torsional modes, for which the group velocity $U=\mathscr{I}_1/C\mathscr{I}_0$ (in JEFFREYS' (1961) notation) is always an excellent approximation to the dispersion $d\omega/dk$ along an overtone $n$=constant.

### 2.3 *What should be called an overtone?*

Going back to the example in Table 2, we achieve a much better agreement between $U$ and $U^*$ by computing $U^*$ along "diagonals" (${}_{50}S_2$–${}_{51}S_3$ and ${}_{51}S_2$–${}_{50}S_3$). As Jeffreys' calculation is, itself, based upon the physical concept of dispersion and uses Eq. (4) as a start, this suggests that the concept of an "overtone branch" as a set of modes sharing some physical property is, in the

Table 2. Physical properties of ${}_nS_1$ ($7 \leq n \leq 62$) and of a few ${}_nS_5$.

| Mode | Period (s) | $U$ (km/s) | $Q_{\mathrm{MMS}}$ | $Q_{\mathrm{SL2}}$ | $y_3$ |
|---|---|---|---|---|---|
| 7 S 1 | 449.2 | 6.82 | 378 | 336 | 5.9638 |
| 8 S 1 | 348.3 | 14.03 | 1,633 | 1,055 | 0.2245 |
| 9 S 1 | 312.3 | 0.67 | 356 | 304 | −2.9751 |
| 10 S 1 | 292.2 | 25.83 | 7,961 | 289 | 0.1600 |
| 11 S 1 | 271.3 | 15.18 | 1,749 | 853 | 0.1306 |
| 12 S 1 | 232.4 | 0.80 | 371 | 290 | 22.8094 |
| 13 S 1 | 222.5 | 14.36 | 1,475 | 934 | −0.1131 |
| 14 S 1 | 201.6 | 25.88 | 7,746 | 296 | −0.0256 |
| 15 S 1 | 188.9 | 16.83 | 1,729 | 816 | 0.0399 |
| 16 S 1 | 186.6 | 0.69 | 333 | 264 | 26.2565 |
| 17 S 1 | 163.4 | 15.55 | 1,732 | 978 | −0.1424 |
| 18 S 1 | 156.1 | 1.18 | 351 | 275 | 8.2746 |
| 19 S 1 | 154.6 | 27.32 | 7,837 | 307 | 0.4659 |
| 20 S 1 | 143.9 | 15.74 | 1,900 | 1,000 | 0.0101 |
| 21 S 1 | 133.1 | 0.33 | 350 | 274 | −8.1449 |
| 22 S 1 | 127.8 | 14.99 | 1,864 | 1,063 | 0.1087 |
| 23 S 1 | 125.9 | 28.07 | 9,881 | 292 | 0.0842 |
| 24 S 1 | 116.6 | 0.34 | 319 | 251 | −65.1630 |
| 25 S 1 | 115.5 | 14.22 | 1,745 | 999 | −0.0428 |
| 26 S 1 | 106.3 | 27.41 | 8,023 | 315 | 0.0441 |
| 27 S 1 | 105.3 | 16.19 | 1,782 | 831 | 0.0777 |
| 28 S 1 | 103.9 | 0.46 | 323 | 251 | −27.5566 |
| 29 S 1 | 97.1 | 16.23 | 1,643 | 970 | −0.0509 |
| 30 S 1 | 93.4 | 0.35 | 329 | 255 | 12.3919 |
| 31 S 1 | 91.9 | 27.68 | 9,120 | 299 | 0.0111 |
| 32 S 1 | 90.2 | 17.38 | 1,768 | 888 | −0.0215 |
| 33 S 1 | 85.0 | 0.42 | 329 | 255 | −8.8121 |
| 34 S 1 | 83.8 | 15.86 | 1,692 | 1,027 | 0.1497 |
| 35 S 1 | 81.0 | 28.31 | 9,936 | 304 | 0.0280 |
| 36 S 1 | 78.4 | 14.69 | 1,767 | 1,017 | −0.0646 |
| 37 S 1 | 77.9 | 0.30 | 324 | 251 | 49.3724 |
| 38 S 1 | 73.4 | 15.55 | 1,852 | 1,010 | 0.0443 |
| 39 S 1 | 72.4 | 28.29 | 9,878 | 297 | 0.1169 |
| 40 S 1 | 71.9 | 0.24 | 324 | 250 | −45.6021 |
| 41 S 1 | 69.1 | 15.96 | 1,700 | 992 | −0.0278 |
| 42 S 1 | 66.8 | 0.21 | 323 | 252 | 22.0528 |
| 43 S 1 | 65.6 | 20.42 | 2,269 | 572 | −0.0242 |
| 44 S 1 | 65.4 | 25.47 | 4,041 | 379 | −0.0262 |
| 45 S 1 | 62.4 | 1.95 | 361 | 278 | −2.9868 |
| 46 S 1 | 62.2 | 14.90 | 1,102 | 735 | 0.3798 |
| 47 S 1 | 59.7 | 28.35 | 9,674 | 297 | −0.0122 |
| 48 S 1 | 59.2 | 15.34 | 1,597 | 943 | −0.0373 |
| 49 S 1 | 58.5 | 0.21 | 334 | 254 | 156.7702 |
| 50 S 1 | 56.4 | 14.96 | 1,677 | 1,024 | 0.0333 |
| 51 S 1 | 55.0 | 0.15 | 340 | 257 | −41.3296 |
| 52 S 1 | 54.9 | 28.55 | 9,940 | 303 | −0.1273 |
| 53 S 1 | 53.9 | 15.16 | 1,648 | 1,007 | −0.0003 |

Table 2. (continued)

| Mode | Period (s) | $U$ (km/s) | $Q_{MM8}$ | $Q_{SL2}$ | $y_3$ |
|---|---|---|---|---|---|
| 54 $S$ 1 | 51.9 | 0.15 | 335 | 256 | 22.6772 |
| 55 $S$ 1 | 51.6 | 17.46 | 1,778 | 962 | −0.0492 |
| 56 $S$ 1 | 50.8 | 28.52 | 9,775 | 297 | −0.0327 |
| 57 $S$ 1 | 49.5 | 17.24 | 1,653 | 992 | −0.1198 |
| 58 $S$ 1 | 49.2 | 0.35 | 337 | 256 | 9.8431 |
| 59 $S$ 1 | 47.6 | 16.50 | 1,710 | 966 | −0.0267 |
| 60 $S$ 1 | 47.3 | 28.65 | 9,595 | 302 | −0.0466 |
| 61 $S$ 1 | 46.8 | 0.16 | 349 | 259 | −267.0852 |
| 62 $S$ 1 | 45.7 | 15.03 | 1,665 | 1,033 | 0.0305 |
| 31 $S$ 5 | 82.7 | 23.96 | 5,645 | 310 | 0.0886 |
| 32 $S$ 5 | 79.1 | 14.79 | 1,737 | 921 | 0.0177 |
| 33 $S$ 5 | 77.6 | 0.77 | 328 | 253 | −1.5586 |
| 34 $S$ 5 | 74.4 | 16.35 | 2,051 | 667 | 0.0281 |
| 35 $S$ 5 | 73.4 | 24.47 | 5,174 | 338 | 0.0347 |
| 36 $S$ 5 | 71.7 | 0.99 | 328 | 253 | 7.5008 |
| 37 $S$ 5 | 69.7 | 14.83 | 1,632 | 968 | −0.0380 |
| 38 $S$ 5 | 66.6 | 1.08 | 329 | 254 | 3.5322 |
| 39 $S$ 5 | 66.4 | 24.85 | 6,362 | 307 | 0.0619 |
| 40 $S$ 5 | 66.0 | 15.90 | 1,671 | 807 | 0.0034 |

present case, better applied diagonally than along lines of constant $n$. Similarly, BRUNE (1964) has shown that the group velocity of a mode can be interpreted in terms of the spatial variation of the phase spectrum of body waves to which this mode contributes. His formalism, however, involved taking derivatives of the phase along overtones branches ($l$=constant in his notation), whose members may not contribute to the same continuous set of body waves in the classical nomenclature.

By doing so, we also regroup modes having comparable values of all physical properties, such as $Q$, and $y_3$ (see Table 1). Also, Fig. 3 shows a plot of the excitation coefficients $N_0$, $K_0$, $K_1$, $K_2$ as a function of depth for each of these four modes. It is evident that there exist a strong correlation between the eigenfunctions of ${}_{50}S_2$ and ${}_{51}S_3$, as well as between those of ${}_{50}S_3$ and ${}_{51}S_2$, rather than along the lines $n=50$ or $n=51$. The modes with large $K_0$ and $N_0$ coefficients are those with larger group velocities, higher $Q$'s and lower $|y_3|$. The other two modes exhibit large $K_1$'s, have $|y_3|$ larger than 1, and share low group velocities and low $Q$'s. From this evidence, it is concluded that mode branches should be allowed to cross, if they are to carry a physical meaning. This point is important, since both the concept of a wave, and the applicability of asymptotic expansion techniques are dependent upon the ability to deal with a whole set of modes whose properties vary continuously with wavenumber of frequency.

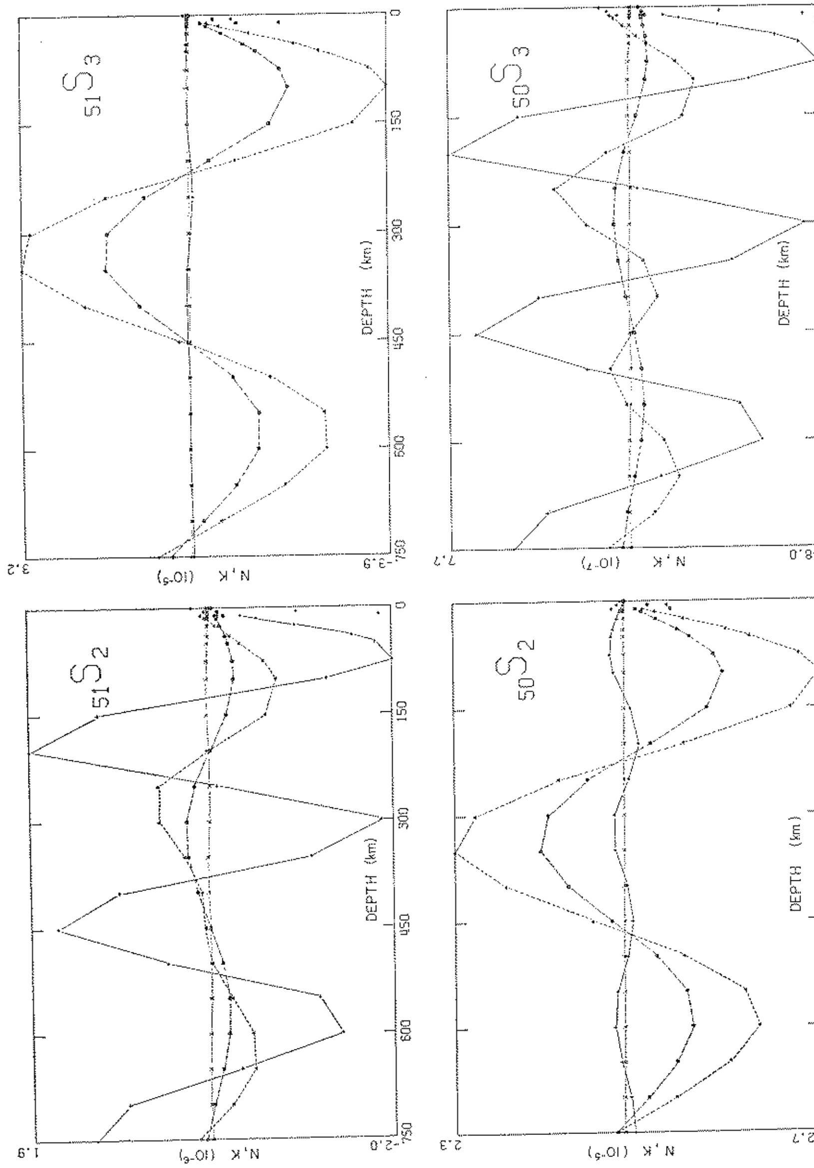


Fig. 3. Excitation coefficients $N_0$, $K_0$, $K_1$, $K_2$ as a function of depth for the four adjacent modes ${}_{50,51}S_{2,3}$. Full lines are $K_1$, dashed lines $K_0$, dotted lines $K_2$ and dash-dot lines $N_0$. For clarity, the functions are not traced in the vicinity of the surface, where strong variations in elastic moduli give them an irregular behavior. Note the "diagonal" correlation between ${}_{50}S_2$ and ${}_{51}S_3$, ${}_{51}S_2$ and ${}_{50}S_3$.

### 2.4 *Empirical evidence for three different families of spheroidal modes at low l*

Table 2 lists the physical properties ($T$, $U$, $Q$, $y_3$) for all modes ${}_nS_1$ ($7 \leq n \leq 62$) and for a few ${}_nS_5$ modes. (The purpose of incorporating the latter is to show that the trends defined by the ${}_nS_1$'s are indeed present for other values of $l$.) From these values, it is easy to identify three families, whose properties are summarized in Table 3.

*1st family*: *K modes.* These modes are characterized by: i) very large group velocities, usually in excess of 25 km/s; ii) values of $Q$ extremely dependent upon the mean $Q$ at the core-mantle boundary (CMB) and the inner core (around 8,000 when using MM8; 300 when using SL2); iii) a low value of $|y_3|$ (on the order of 0.1); and iv) very low excitation coefficients for all depths down to 750 km. Items ii) and iv) clearly identify these modes as Inner Core modes (we use the letter $K$ from the German "Kern"): The attenuation model SL2 (ANDERSON and HART, 1978) is characterized by a high-attenuation layer at the base of the mantle and in the inner core. It can also be proved that $K$ modes are indeed the unobserved "core" modes, as defined by GILBERT and DZIEWONSKI (1975).

*2nd family*: *C (Colatitudinal) modes.* This is a family of highly attenuated modes, with low group velocities, and high values of $|y_3(a)|$. ${}_{51}S_2$ and ${}_{50}S_3$, studied in the previous section, belong to this family. Furthermore, Fig. 2 shows, in the case of these two modes, that at all depths the excitation function $K_1$ remains very large, with $K_0$ still substantial, and about 10 times as large as $K_2$. This property, illustrated in the case of this particular example, is indeed a common factor of the family. In view of the expressions for the coefficients $K$ (KANAMORI and CIPAR, 1974), this suggests that, for these modes, the function $|y_1(r)|$ remains small with respect to $|y_3(r)|$ at all depths. It is clear that the displacement in these modes is mainly colatitudinal, hence the "$C$".

*3rd family*: *V (Vertical) modes.* This family has intermediate group velocities (10 to 18 km/s for ${}_nS_1$), high values of $Q$ in both models MM8 (1,000–

Table 3. Properties of the different classes of modes for low $l$.

| Property | $K$ | $C$ | $V$ |
|---|---|---|---|
| Group velocity $U$ (km/s) | 20–30 | 0.3–4 | 9–17 |
| $Q$ | SL2: 300<br>MM8: 4,000–9,000 | 200–400 | 700–2,000 |
| $y_3$ | Low<br>(down to $10^{-3}$) | $>1$<br>Up to 100 | $<1$<br>Down to 0.3 $10^{-2}$ |
| Excitation coeffs. | Low | $K_1$ substantial<br>$K_0$ comparable to $K_1$ | Low $K_1$ and $K_2$<br>Substantial $K_0$ and $N_0$ |

Table 4. Two examples of coupled spheroidal modes at low $l$.

| Mode | Period (s) | $U$ (km/s) | $Q_{MM8}$ | $Q_{SL2}$ | $y_3$ |
|---|---|---|---|---|---|
| ${}_{20}S_3$ | 129.8 | 19.45 | 1,858 | 458 | 0.18 |
| ${}_{21}S_3$ | 127.7 | 20.41 | 2,972 | 398 | 0.11 |
| ${}_{15}S_4$ | 158.0 | 8.84 | 720 | 541 | −0.28 |
| ${}_{16}S_4$ | 152.3 | 7.55 | 494 | 390 | 0.39 |

4,000) and SL2 (700–2,000), and very small values of $|y_3(a)|$. Furthermore, as shown on Fig. 3, and confirmed by a further study, these modes have large values of $N_0$ and $K_0$, and very small $K_1$ and $K_2$. All of this suggests that the displacement in these modes is mainly radial or "vertical", hence the "$V$". These properties are shared by the radial modes ${}_nS_0$, which are in the same number (20 modes from 200 s. down to 45 s.), and have, by definition, no colatitudinal displacement or excitation functions $K_1$ and $K_2$. The radial modes are thus part of the $V$ family.

At this point, we have defined empirically three different families of modes with low $l$, clearly identified only for phase velocities larger than about 26 km/s. There happen isolated cases, when the individual properties of two modes may violate in some respect the general trends in their families. Table 4 gives two examples. We will see later that these are clear cases of coupling, due to a near coincidence in eigenfrequencies. The fundamental point is that, in this region of the $(\omega, l)$ plane ($C>26$ km/s), these are *isolated occurences*, which do not represent any general physical trend.

*2.4.1 Modes with larger values of $l$*

For low values of the phase velocity ($C<16$ km/s), Fig. 1 shows that spheroidal modes are arranged along well-defined, regular, overtone branches. As in the case of torsional modes, the physical properties of the modes vary smoothly along those branches, and regularly from one branch to the next. We will call this family of modes "$R$" (Rayleigh) modes: The fundamental $R$ modes do indeed generate classical Rayleigh waves, and it can be shown that higher overtones are similar to Rayleigh wave overtones, as described, for example, by HARKRIDER (1970).

At values of $l$ for which the phase velocity falls in the range 16–26 km/s, there is no such definite behavior. It would indeed be possible to try to define two groups of branches (as shown in the upper part of Fig. 8) and associate them with $V$ or $C$ modes, or to allow large undulations along branches (as in the bottom of Fig. 8), and make them $R$ modes. However, neither of these approaches is satisfactory, since, in both of them, physical properties do not remain constant along the branches. We will call these modes "$H$" (Hybrid) modes.

The three families of modes $K$, $C$, $V$, isolated empirically at low $l$ in this section, are, indeed, the ones defined theoretically by ANDERSSEN *et al.* (1975) and GILBERT (1975). However, their theoretical investigations were limited primarily to the periods $T$ of the modes, and to the general character (vertical, colatitudinal or core) of the solution. Also, they only made use of the extreme limiting case $l=0$ (or $l=1$ in the case of $K$ and $C$ modes) in their comparison with actual computed or observed values, although the three families can still be identified at higher values of $l$. (Table 2 lists a few modes with $l=5$.) ANDERSSEN *et al.* (1975) and GILBERT (1975) did not study the group velocity of the modes, which is characteristic of their variation with $l$, nor $Q$, nor the excitation functions. These properties are of fundamental importance in any attempt to synthetize seismograms by asymptotic mode theory. In the next section, we will show that it is possible to derive the group velocity of the families of low-$l$ modes ($K$, $C$, $V$) and to extend most of the properties derived by ANDERSSEN *et al.* (1975) to non-zero values of $l$, under very simple, if somewhat crude, assumptions.

## 3. *A Theoretical Approach to the Various Properties of Spheroidal Modes of Low l*

In order to show that most of the properties of the various families of modes can be derived simply, we shall consider here the normal modes of a homogeneous sphere (with only the possibility of a fluid core), and we shall neglect the influence of gravity.

The theoretical problem of the spheroidal eigenvibrations of a homogeneous sphere was studied by LOVE (1911) and PEKERIS and JAROSCH (1958). TAKEUCHI and SAITO (1972) gave the comple solution for the eigenfunctions of a homogeneous sphere (pp. 243–244). The reader is referred to this paper for the exact (and rather elaborate) expressions of the solution. Some of the theoretical results in the following section were given by ANDERSSEN *et al.* (1975) and GILBERT (1975) for constant $l$ (mostly $l=0$ and $l=1$). Their extension to a variable $l$ allows a theoretical study of the group velocity of the modes.

Rather than start from the exact solutions and adapt the equations to our particular cases of interest, we will try to simplify the system of differential equations for spheroidal modes, before solving it, by use of physical arguments, thus keeping a stronger physical insight into the properties of the solutions. We start with SAITO's (1967) system, and we assume that we can neglect the influence of gravity. The system then reduces to: $\mathrm{d}\mathbf{y}/\mathrm{d}r=C\mathbf{y}$, with $\mathbf{y}^T=[y_1, y_2, y_3, y_4]$, and:

$$C=\begin{bmatrix} -\frac{2\lambda}{\lambda+2\mu}\cdot\frac{1}{r} & \frac{1}{\lambda+2\mu} & \frac{L^2\lambda}{r(\lambda+2\mu)} & 0 \\ -\omega^2\rho+\frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} & -\frac{4\mu}{\lambda+2\mu}\cdot\frac{1}{r} & -L^2\frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} & \frac{L^2}{r} \\ -\frac{1}{r} & 0 & \frac{1}{r} & \frac{1}{\mu} \\ -\frac{2\mu}{r^2}\cdot\frac{3\lambda+2\mu}{\lambda+2\mu} & -\frac{\lambda}{\lambda+2\mu}\cdot\frac{1}{r} & -\omega^2\rho-\frac{2\mu}{r^2}+4L^2\cdot\frac{\mu(\lambda+\mu)}{(\lambda+2\mu)r^2} & -\frac{3}{r} \end{bmatrix} \quad (5)$$

Here again, $L^2=l(l+1)$. For $l=0$, this matrix has an entire quadrant of zeroes, which means that the system breaks down into two completely independent systems of order two: one involving $y_1$ and $y_2$, the other $y_3$ and $y_4$. We will now solve both of these simple systems, and look at the case of small, but non-zero $L^2$.

### *3.1 Solutions with mainly radial displacement ("V" modes)*

For $l=0$, the system is completely decoupled, and, as shown by GILBERT (1975), the solution for a homogeneous sphere compatible with the boundary conditions is a spherical Bessel functions of order 1:

$$y_1=j_1(\omega r/\alpha)\,. \quad (6)$$

For high overtones:

$$_p\omega_0\simeq p\pi\alpha/a\,. \qquad (p \text{ integer}) \quad (7)$$

The frequency spacing between the subsequent radial modes $_nS_0$ is therefore $\Delta f=\alpha/2a$. The average value of $\alpha$ in the Earth is 10.46 km/s (JEFFREYS and BULLEN, 1940). This yields $\Delta f=8.21\cdot10^{-4}$ Hz, in excellent agreement with the average separation observed for $_nS_0$: $\Delta f=8.27\cdot10^{-4}$ Hz. These results are similar to those of ANDERSSEN *et al.* (1975).

For $l$ small, but nonzero, it is no longer possible to ignore $y_3$ and $y_4$: The third column in matrix (5) will inevitably bring coupling between the vertical and horizontal motions. We will therefore simply assume that $|y_3|\ll1$, and that the corresponding modes are still basically irrotational, that is:

$$\mathrm{curl}\,\boldsymbol{u}=\boldsymbol{O}\,. \quad (8)$$

Using (5), this yields:

$$y_3'=\mathrm{d}y_3/\mathrm{d}r=-(y_3-y_1)/r\simeq y_1/r\,. \quad (9)$$

This shows that while $y_3$ might be negligible, its derivative $y_3'$ should be kept in all equations. Then, from the third line of (5), $y_4=2\mu y_1/r$, and eliminating $y_2$ from the first two lines yields:

$$y_1''+\frac{2}{r}y_1'+\left[\frac{\omega^2}{\alpha^2}-\frac{L^2+2}{r^2}\right]y_1+\left[2+\frac{2\mu(\lambda-2\mu)}{(\lambda+2\mu)^2}\right]y_3/r^2=0\,. \tag{10}$$

Given that $|y_3|\ll|y_1|$, and that for the actual Earth, either $\mu=0$ (in the core), or $\mu=\lambda$ (in the mantle), we can approximate the last bracket by 2, and then, bearing in mind that $(ry_3)'=y_1$, we find that this equation is nothing but the derivative of the spherical Bessel equation of order $l$, written for the function $(ry_3)$. We conclude that the solution for $y_1$ is:

$$y_1(r)=j_l'(\omega r/\alpha)\,. \tag{11}$$

Note that this solution holds for $l=0$, since $j_0'=-j_1$. Using the well-known expressions (Abramowitz and Stegun, 1972 (p. 324)):

$$j_\nu(x)=\sqrt{\pi/2x}\cdot J_{\nu+1/2}(x) \qquad \text{and} \qquad J_\nu(x)\simeq\sqrt{2/\pi x}\cdot\cos\,(x-\nu\pi/2-\pi/4) \quad \text{for} \quad x\gg1\,, \tag{12}$$

we find that the boundary condition $y_2(a)=0$ defining the angular eigenfrequencies of the $V$ modes will rapidly require:

$$_p\omega_l=\frac{\alpha}{a}\pi[p+l/2]\,. \tag{13}$$

Here, $p$ is a new "overtone" number inside the $V$ family. We have thus derived the law of variation of the eigenfrequencies $_p\omega_l$ of the $V$ modes, both with $p$ and $l$. This relation has two consequences.

*3.1.1 The frequency spacing at a given l should be independent of l*

This is checked against a realistic Earth model in Table 5: The frequency spacing varies less than 2% from $l=0$ to $l=9$, and stays within 3% of the value computed on the basis of our rather crude assumptions. This result was implicit from Gilbert's paper. However, the numerical values had only

Table 5. Comparison of theoretical and observed values of frequency spacing and group velocity of the three families $V$, $K$, $C$.

| | $V$ | $K$ | $C$ |
|---|---|---|---|
| $\Delta f$ (Hz) computed: | $8.21\cdot10^{-4}$ | $1.43\cdot10^{-3}$ | $1.07\cdot10^{-3}$ |
| $\Delta f$ (Hz) observed: | | | |
| $l=0$ | $8.27\cdot10^{-4}$ | | |
| $l=1$ | $8.29\cdot10^{-4}$ | $1.47\cdot10^{-3}$ | $1.07\cdot10^{-3}$ |
| $l=5$ | $8.32\cdot10^{-4}$ | $1.50\cdot10^{-3}$ | $1.06\cdot10^{-3}$ |
| $l=9$ | $8.41\cdot10^{-4}$ | $1.54\cdot10^{-3}$ | $1.05\cdot10^{-3}$ |
| $U^*$ (km/s) computed: | 15.7 | 28.6 | 0 |
| $U$ (km/s) observed: | | | |
| $l=1$ | 15.82 | 27.56 | 0.50 |
| $l=5$ | 13.94 | 24.43 | 1.80 |
| $l=9$ | 13.74 | 23.01 | 3.31 |

been checked against the theory by ANDERSSEN *et al.* (1975) for $l=0$ and $l=1$.

*3.1.2 The dispersion group velocity along a branch of V modes of constant p can be predicted*

According to (13), we have:

$$d\omega/dk = a \qquad d\omega/dl = \alpha\pi/2 . \tag{14}$$

The group velocity of a $V$ branch should be $\pi/2$ times the average Earth's $P$-wave velocity. Again, taking the latter as 10.46 km/s, we obtain a figure of 16.43 km/s, in excellent agreement with the values of $U$ characteristic of $V$ modes. This agreement between $U$ and $d\omega/dk$ along a $V$ branch confirms that a branch of $V$ modes with constant $P$ is, indeed, a set of physically continuous modes. A further consequence is that the following relation holds:

$$_{p+1}\omega_l \simeq {}_p\omega_{l+2} . \tag{15}$$

This fact is confirmed in Table 6 and in the general layout of $V$ modes on Fig 6.

It is important to note that the radial modes $_pS_0$ completely share these properties, and can therefore be totally integrated into the $V$ family.

The physical property limiting the field of separation between $K$, $C$ and $V$ modes in the $(\omega, l)$ plane is the phase velocity $C$. Therefore, for high overtones, one expects the characteristic properties of the various families to hold even for relatively large values of $l$, ($l=10$–$20$), for which the asymptotic expansion of the Legendre associated functions, as suggested by KANAMORI and STEWART (1976), is valid and justifies the use of asymptotic theory. It is

Table 6. Examples of the law $_p\omega_{l+2} \simeq {}_{p+1}\omega_l$ for $V$ and $K$ modes.

| Mode in conventional new nomenclature | | Period (s) | Error |
|---|---|---|---|
| $_{10}S_0$ | $_{10}V_0$ | 110.4 | |
| $_{25}S_2$ | $_9V_2$ | 110.8 | 0.3% |
| $_{23}S_4$ | $_8V_4$ | 111.8 | 0.9% |
| $_{21}S_6$ | $_7V_6$ | 113.0 | 1.1% |
| $_{18}S_8$ | $_6V_8$ | 116.2 | 2.8% |
| $_{38}S_{15}$ | $_{14}V_{15}$ | 56.2 | |
| $_{36}S_{17}$ | $_{13}V_{17}$ | 56.5 | 0.6% |
| $_{34}S_{19}$ | $_{12}V_{19}$ | 57.3 | 1.6% |
| $_{32}S_{21}$ | $_{11}V_{21}$ | 57.8 | 1.0% |
| $_{39}S_1$ | $_{10}K_1$ | 72.4 | |
| $_{37}S_3$ | $_9K_3$ | 72.7 | 0.4% |
| $_{35}S_5$ | $_8K_5$ | 73.4 | 1.0% |
| $_{32}S_7$ | $_7K_7$ | 75.1 | 2.3% |
| $_{30}S_9$ | $_6K_9$ | 76.6 | 2.0% |

therefore fundamental to have established a theoretical proof of the validity of the group velocities $U$, computed from Jeffreys' formulas, as an accurate representation of the physical dispersion along a $V$ branch.

*3.2 Solutions with mainly colatitudinal displacements* (*C modes*)

For $l=0$, there can be no physical solution, since they would generate no displacement. However, this limiting case will be helpful when $L^2 \neq 0$. Assuming that $y_1 = y_2 = 0$, and following GILBERT (1975), one finds that the solution $y_3$ now involves both the Neumann and Bessel spherical functions of order 0. The solution for $\omega$ is rapidly Gilbert's Eq. (8):

$$ {}_p\omega_0 = p\frac{\pi\beta}{a-r_c}, \tag{16} $$

in total similarity to the torsional case. This yields $\Delta f = \beta/2(a-r_c) = 1.07 \cdot 10^{-3}$ Hz, for an average $S$-wave velocity of 6.18 km/s (JEFFREYS and BULLEN, 1940).

For $l$ nonzero, it is no longer possible to assume $y_1 = 0$. Similarly to our study of $V$ modes, we will assume that $\operatorname{div} \boldsymbol{u} = 0$, and that $|y_1| \ll |y_3|$. Then, we have, from (5):

$$ y_1' = L^2 y_3 - 2y_1/r \simeq L^2 y_3 , \tag{17} $$

and

$$ y_3'' + \frac{2}{r} y_3' + \left[\frac{\omega^2 \rho}{\mu} - \frac{L^2}{r^2}\right] y_3 = 0 . \tag{18} $$

Technically, this is a Bessel equation of order $l$. However, since $r$ is not allowed to become smaller than the radius of the core, $r_c$, then $L^2/r^2$ remains smaller than $L^2/r_c^2$. As long as we have

$$ \omega^2 \gg \frac{\beta^2}{r_c^2} \cdot L^2 , \quad \text{or roughly} \quad C \gg \beta \cdot a/r_c \simeq 12 \text{ km/s} , \tag{19} $$

this new term will be negligible, and the solution (whose boundary conditions are unchanged) will remain very similar to that for $l=0$. Also, as $r$ never becomes zero, the coupling coefficients of order $L^2/r$, in the matrix (5) will remain very small, and ensure that $y_1$ and $y_2$ are small. This is a fundamental difference which makes $V$ and $C$ modes behave differently. In other words, since the wave cannot propagate into the core, and as long as it has a high enough frequency, it cannot "feel" the curvature of the Earth; therefore, it is insensitive to the angular order number $l$, which characterizes the variation with $\theta$, and behaves like a plane wave between two fluid boundaries: This is what is expressed by Eq. (20):

$$ {}_p\omega_l = {}_p\omega_0 = \frac{p\pi\beta}{(a-r_c)} . \tag{20} $$

An entirely similar situation would arise with a torsional system, and therefore, both $T$ and $C$ modes should be very similar to their common limit when $L^2 \to 0$, and share the following properties: $\Delta f = \beta/2(a-r_c) = 1.07 \cdot 10^{-3}$ Hz, and a very small group velocity at low $l$. Both of these properties are checked in Table 5. As can be seen, the agreement extends for values of $l$ larger than 1. A complete similarity exists between solutions for $C$ and $T$ modes (except for a different normalization of the eigenfunctions), holding for periods, group velocities, $Q$'s, and excitation functions, $K_1$, $K_2$, $L_1$ and $L_2$, although $C$ modes also have substantial $K_0$ coefficients, making a possible contribution to the seismic displacements, especially at nearby vertical incidence (BULAND, 1977 and personal communication).

The identity between $C$ and $T$ modes is the mode-theory aspect of the trivial identity between SH and SV waves at vertical incidence. The absence of systematic coupling between $C$ and $T$ modes at high phase velocities simply expresses the vanishing of all P–SV transmission and reflection coefficients at zero incidence.

### *3.3 Inner core (K) modes*

Equation (18) can be applied not only to the mantle, but also to the solid inner core, although neglecting gravity certainly becomes a much poorer approximation. However, the interest in core modes is, at the present time, purely theoretical, as none of them is significantly excited by any realistic seismic source. Boundary conditions now require $y_3(0)=0$; $y_4(r_i)=0$, where $r_i$ is the radius of the inner core ($r_i = 1{,}215$ km). The solution is simply:

$$y_3 = j_l(\omega r/\beta)\,, \tag{21}$$

which leads to eigenfrequencies:

$${}_p\omega_l = \frac{\beta}{r_i}\pi\left[p + \frac{l}{2} + \frac{1}{2}\right]. \tag{22}$$

This formula extends GILBERT's (1975) Eq. (9) by incorporating variations of $l$. The frequency spacing between $K$ modes should then be $\Delta f = \beta/2r_i = 1.43 \cdot 10^{-3}$ Hz, for an average shear velocity of 3.48 km/s inside the inner core (ANDERSON and HART, 1976), and the group velocity theoretically predicted at the surface of the Earth: $U = \beta a\pi/2r_i = 28.6$ km/s. These values are indeed characteristic of $K$ modes, as is checked in Table 5. Also, $K$ modes should follow the law ${}_p\omega_{l+2} \simeq {}_{p+1}\omega_l$; Table 6 shows that this relation holds to within a few percent. The low value of $y_3(a)$ associated with $K$ modes is simply a consequence of the inability of the outer core to transmit transverse displacements: only the small vertical component $y_1$ can be transmitted through it to the mantle and surface of the Earth.

Although the displacement in them is similar ($|y_3| \gg |y_1|$), Eq. (20) and (22) show that $C$ and $K$ modes behave totally differently with varying $l$, a fact which is not explicitly brought out in Gilbert's approach. When $l$ varies, $K$ and $V$ modes do indeed share a similar behavior, as their general layout shows on Figs. 4 and 6. It can be said that *they* are the modes of a spherical body. As shown in Section 3.2, the mantle's $C$ modes, insensitive to $l$, behave like plane waves.

In summary, by assuming that the original system is entirely decoupled at low $l$, we have been able to derive most of the properties of the three families of modes, and their variation with $l$. Note that the various assumptions which were made on $|y_3/y_1|$, curl $\boldsymbol{u}$, etc. can be *a posteriori* checked to hold: It is possible to do so by using an exact solution, such as the one in TAKEUCHI and SAITO (1972, p. 243), and assuming (in their notation) $\gamma=0$; the upper sign in their Eq. (99) corresponds to $V$ modes, the lower one to $C$ modes.

### *3.4 Coupling at larger l*

We will not discuss the large-$l$ limit of the differential system (5), since the theoretical study of surface waves has been quite extensive in the past decades. It can be shown very easily, that, at large $l$, (5) reduces to the classical equations governing surface waves (TAKEUCHI *et al.*, 1962; SAITO, 1967). However, we would learn nothing about overtones from the crude model we started with, since at high frequencies, when the Earth's curvature can be neglected, a homogeneous medium yields only the fundamental Rayleigh wave ${}_0S_l$.

On the other hand, it can be useful to examine the coupling between $V$ and $C$ modes as a function of $l$ as a particular case of coupling between two dynamical systems. This is an extremely frequent phenomenon in Physics, and in all fields (quantum mechanics, solid state physics, oscillatory dynamics, electrodynamics...), its basic effects (spreading out the eigenfrequencies and hybridizing the eigenfunctions) are essentially similar. Examples can be found in most textbooks (e.g. ROCARD, 1948; KITTEL, 1963). In order to analyze the different forms of coupling occurring at various ranges of $l$, let us think of the Earth as a physical system having two linear unperturbed dispersion relations: $\omega=\omega_c$ ($C$ modes), and $\omega=\omega_0+U\cdot k$ ($V$ modes), where $k$ is the wavenumber around the surface of the Earth, $U$ is the group velocity of the corresponding $V$ modes. Although we have seen that the exact mathematical formulation of the problem is more intricate, the physical problem can be schematized by allowing the dispersion relation to become:

$$(\omega-\omega_c)(\omega-\omega_0-U\cdot k)=\varepsilon^2\,, \tag{23}$$

where $\varepsilon$ is some measure of the intensity of coupling. As shown on Fig. 10, the dispersion curves are hybridized into the two branches of a hyperbola:

$$\omega = 1/2[\omega_0 + U\cdot k + \omega_e] \pm 1/2\sqrt{(\omega_0 + U\cdot k - \omega_e)^2 + 4\varepsilon^2}\,. \tag{24}$$

It is clear from Fig. 10 that the range $\Delta k$ of wavenumbers over which the hybridization takes place increases with $\varepsilon$, and it can be shown from Eq. (24) that

$$\Delta k = \frac{1}{\sqrt{A}}\frac{\varepsilon}{U}\,, \tag{25}$$

where $A$ is some measure of the maximum allowed perturbation of a branch.

Now, in the present case of the Earth, the coupling terms in Eq. (5) are proportional to $L^2$. However, the physical quantities directly comparable are the displacements $u_r$ and $u_\theta$. Expressions from SAITO (1967) show that $u_r \propto y_1\cdot P_l^m(\theta)$, while $u_\theta \propto y_3\cdot \mathrm{d}P_l^m(\theta)/\mathrm{d}\theta$. Therefore, for $l \gg 1$, $|u_\theta/u_r| \simeq (l+1/2)\cdot|y_3/y_1|$, and the physical intensity of the coupling is really proportional to $l$. Furthermore, wavenumbers $k$ can take only discrete values: $k=(l+1/2)/a$, separated by $\delta k = 1/a$. Then:

i) *At low l*

The hybridization occurs over a very small range $\Delta k$, smaller than the unit interval $\delta k$, and coupling goes unnoticed; only in the unfavorable case when a discrete value of $k$ falls in the close vicinity of the crossing point of two branches, do we observe any hybridization of the modes. These are the few anomalous modes observed within the $V$ and $C$ families. It is also interesting to note that slight departures from the properties of a family, as found in Tables 2, 4, 6 are an indication of such a circumstance. This translates into a bump in the curves on both Figs. 5 and 6. A similar situation arises between ${}_{10}S_2$ and ${}_{11}S_2$ (${}_3V_2$ and ${}_3K_2$), leading to the possible excitation of the latter by an earthquake, despite its nature as a core mode. The observation of this occurrence, by DZIEWONSKI and GILBERT (1973) proved the existence of $K$ modes, and demonstrated the solidity of the inner core.

ii) *At intermediate l*

$\Delta K$ is on the order of a few sampling units $\delta k$, and the two hybridized branches can be continuously identified: The properties of the modes along them vary continuously from one type of family to the other. However, the coupling range $\Delta k$ is still small enough that multiple coupling between more than two overtone branches does not take place: This is the general behavior of $H$ modes.

iii) *At large l*

$\Delta K \gg \delta k$, the hyperbola degenerates into two parallel straight lines, and the modes are totally hybridized, over the whole spectrum (in fact coupling involves more than two branches at a time). This is of course the case of $R$

modes, giving birth to a surface wave in which vertical and horizontal displacements are completely mixed.

### 4. *A Classification and Proposed New Nomenclature for Spheroidal Modes*

In this section, we give a formal classification of all spheroidal modes used in the present study, which is compatible with their physical properties, as discussed in the previous sections. This classification formally extends the one outlined in ANDERSSEN *et al.* (1975), by identifyidg $R$ and $H$ modes and by providing general, although revisable, bounds on the various families. Modes were computer-sorted by meeting the following requirements:

1) $K$ Modes: $C \geq 26$ km/s and $U \geq 21$ km/s.
2) $C$ Modes: $C \geq 26$ km/s, $U < 8$ km/s and $Q_{\mathrm{MM8}} < 500$.
3) $V$ Modes: $C \geq 26$ km/s, $8 \leq U < 21$ km/s and $Q_{\mathrm{MM8}} \geq 500$.
4) $R$ Modes: $C < 16.5$ km/s.
5) $H$ Modes: $16.5 \leq C < 26$ km/s.

Except for $H$ modes in general, and apart from a very restricted number of isolated cases of $V$ and $C$ modes, it is found that, as expected, this classification separates modes along "pseudo-overtone" branches with smoothly varying properties. We use the term "pseudo-overtone", or "pseudo-branch" with reference to the study of a similar behavior in the problem of coupled air and sea waves (PRESS and HARKRIDER, 1966; HARKRIDER and PRESS, 1967). In a few cases, in which coupling is important between $V$ and $C$ modes, both of the coupled modes would fall into a given family. A small violation of the above requirements was then allowed to bring the slightly hybridized mode back into its original family. It was also decided to incorporate the two branches of Stoneley modes with phase velocities around 8.5 and 16 km/s into the $K$ family, due to their low excitation coefficients. This helps define the pseudo-overtone branches of the $R$ family. Due to the close coupling between the modes ${}_1S_l$ and ${}_2S_l$ ($l \leq 15$), these modes were included into the $H$ family.

We now introduce a new nomenclature for spheroidal modes which identifies the family and pseudo-overtone branch to which a mode belongs. This nomenclature will use the five letters $K$, $C$, $V$, $H$, $R$ and two indices: $p$ (pseudo-overtone index) and $l$ (angular order index). The following paragraphs discuss the assignment of the index $p$ in the five families. Figures 4 to 8 are frequency vs. angular order plots of the various families. Extensive tables, giving the correspondence between the new nomenclature and the conventional one (taken as resulting from the use of model 1066A (GILBERT and

DZIEWONSKI, 1975) have been prepared, and are available from the author on request.

*4.1 ${}_pK_l$. Core and Stoneley modes. (See Fig. 4)*

The two Stoneley branches at low $C$ are labeled ${}_0K_l$ and ${}_1K_l$. Then, modes with given $l$ are sorted by increasing frequency. These modes are not excited by any realistic seismic source.

*4.2 ${}_pC_l$. Colatitudinal modes. (See Fig. 5)*

The various pseudo-overtone branches ard labeled so as to realize the identity between ${}_pC_l$ and ${}_pT_l$ modes, as $l \to 0$. In this way, there does not exist a ${}_1C$ branch. This point should not appear as a drawback to the new nomenclature: it simply means that the corresponding modes are not part of the $C$ family. In fact, they are $H$ modes. Figure 5 shows that bumps do occur along $C$ pseudobranches, bringing in negative apparent group velocities This fact results from coupling with $V$ modes, as discussed in section 3.4.

*4.3 ${}_pV_l$. Vertical modes. (See Fig. 6)*

Similarly, the various pseudobranches are labeled so as to let ${}_pS_0$ and

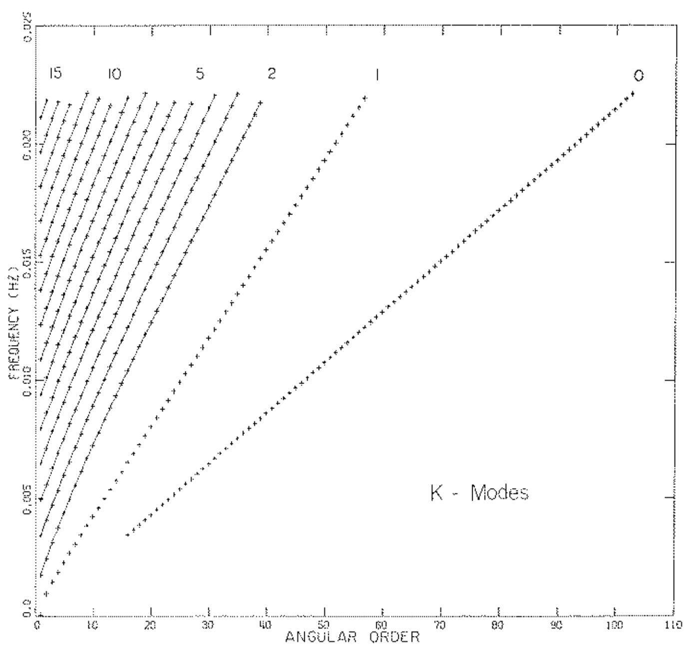


Fig. 4. Frequency vs. angular order plot of the $K$ family of modes. Pseudo-overtones are traced and labeled according to the proposed new nomenclature.

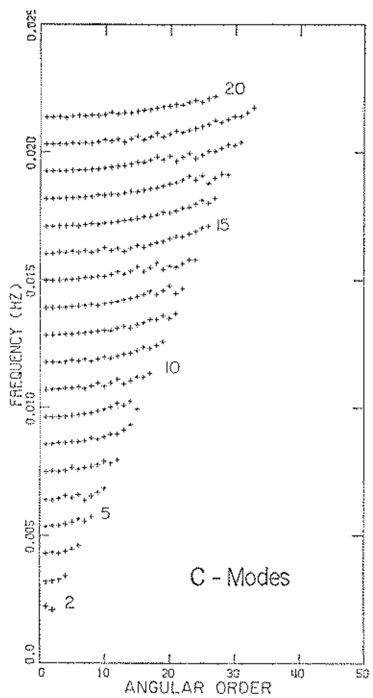


Fig. 5. Frequency vs. angular order plot of the $C$ family of modes. Pseudo-overtones are labeled according to the proposed new nomenclature.

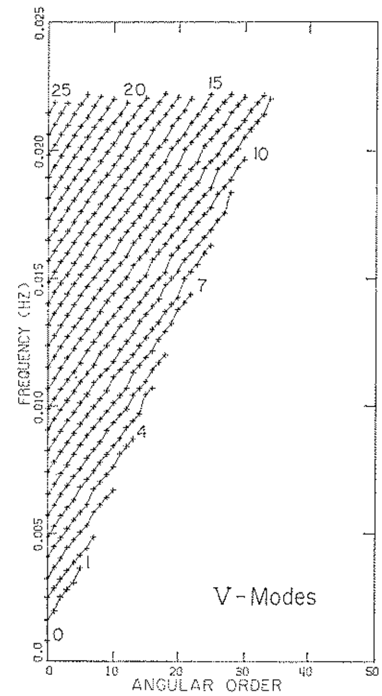


Fig. 6. Frequency vs. angular order plot of the $V$ family of modes. Pseudo-overtones are traced and labeled according to the new nomenclature. Note the easy integration of the radial modes ($l=0$) into the family.

${}_pV_0$ coincide. Again, except for ${}_0V_0$, there are no ${}_0V_l$ modes, and the maximum number for $l$ at a given $p$ is itself a function of $p$. This fact simply means that the missing modes are part of the $H$ family.

*4.4* ${}_pR_l$. *Rayleigh modes. (See Fig. 7)*

The only basic difference in nomenclature between ${}_pR_l$ and ${}_nS_l$ results from the removal of the Stoneley modes ${}_0K_l$.

*4.5* ${}_pH_l$. *Hybrid modes. (See Fig. 8)*

These modes, which correspond to intermediate coupling, lie at a crossing-point between two trends: the decoupled pseudobranch of $V$ and $C$ modes, and the coupled Surface wave trend of $R$ modes. Note that the distinction between $H$ modes and $V$ or $C$ modes, or between $H$ modes and $R$ modes is extremely subjective and depends entirely on the amount of distortion one is willing to allow within the physical properties of $V$, $C$, or $R$ modes, in other

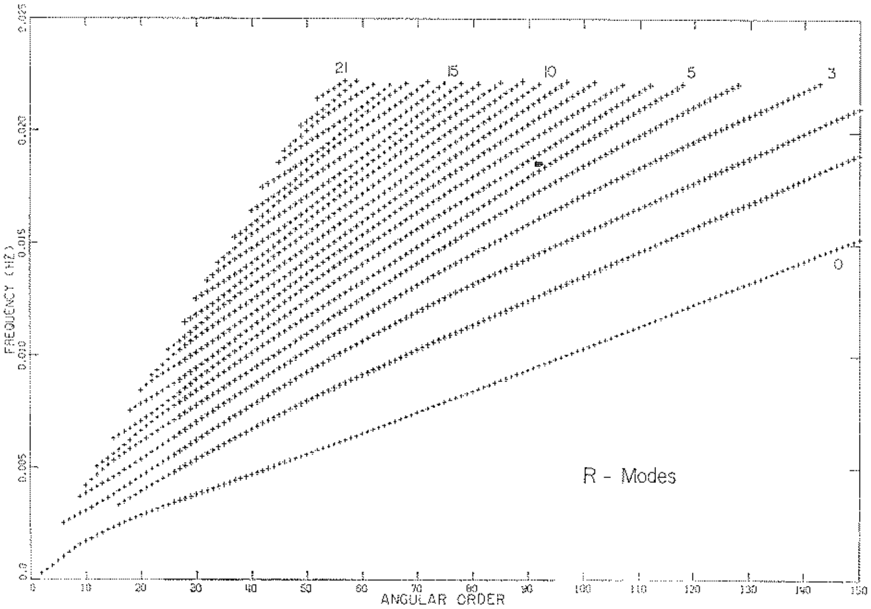


Fig. 7. Frequency vs. angular order plot of the $R$ family of modes. Pseudo-overtones are labeled according to the proposed new nomenclature.

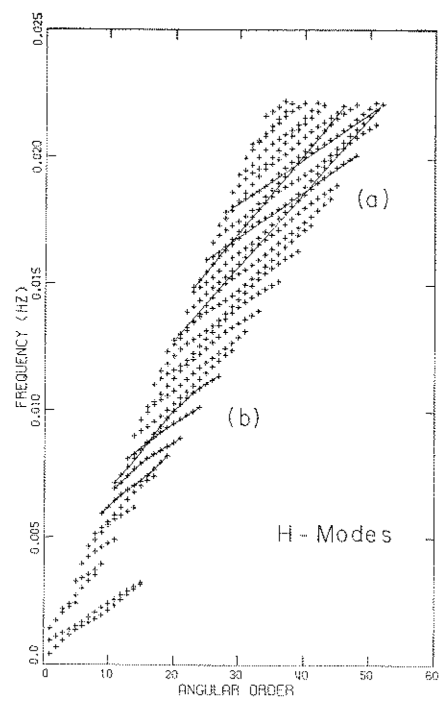


Fig. 8. Frequency vs. angular order plot of the $H$ family of modes. (a) Shows an attempt to neglect coupling and incorporate the modes into the $C$ and $V$ families; (b) Shows the opposite attempt to make them part of the $R$ family.

words, upon the allowance made for hybridization, a quantity similar to $A$ in Eq. (25). Ideally, Eq. (5) shows that, except for $l=0$, coupling between radial and colatitudinal displacements is never totally absent from any mode, and all modes could therefore be considered $H$ modes. This is the basis for the conventional nomenclature, which, however, leads to a loss of most physical insight in the properties of spheroidal modes. We believe that the adopted values (16.5 and 26.5 km/s) for phase velocity bounds on $H$ modes maintain a reasonable balance between the two tendencies. Note that the pseudo-overtone number, $p$, of no $V$, nor $C$, nor $R$ mode, is dependent on those bounds. Should the bounds change, a given mode might be moved out of his family, into another one, but it will retain its $p$ index as long as it stays in-

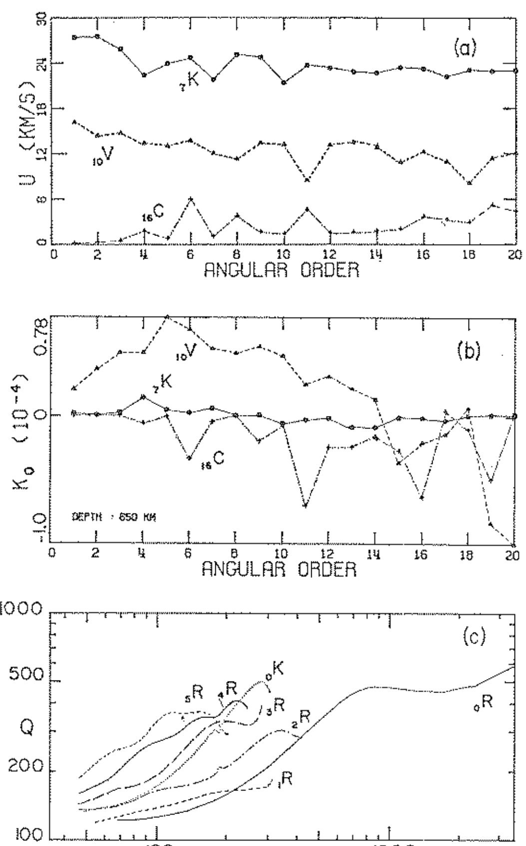


Fig. 9. A plot of physical properties of modes along pseudo-overtone branches. (a) Group velocity vs. angular order along $_7K_l$, $_{10}V_l$ and $_{16}C_l$; (b) Excitation coefficient $K_0$ at 650 km, along $_7K_l$, $_{10}V_l$, $_{16}C_l$; (c) $Q$ along $_0R_l$–$_5R_l$ and $_0K_l$, as a function of period.

side a given family. Given this evidence, and in order to achieve the same stability for $H$ modes, it appears that the only reasonable nomenclature for $H$ modes is to retain the conventional overtone number, that is to simply have ${}_nS_l$ relabeled ${}_nH_l$.

We can now check the effect of the new nomenclature on the five major drawbacks mentioned in section 2.2:

i) Once the values of $p$ are used for both vertical ${}_pV_l$ and colatitudinal ${}_pC_l$ modes, and in the absence of occasional coupling, the eigenfunction ($y_1$ for $V$ modes, $y_3$ for $C$ modes) has $p$ zero-crossings inside the Earth. In the case of the simple system described in section 3, this follows from the properties of the spherical Bessel functions. In the case of a radially heterogeneous Earth, this result comes from the Sturm-Liouville nature of the uncoupled differential system, either for $V$ or $C$ modes. (INCE, 1956; p. 233).

ii) We have already shown that the radial modes ${}_nS_0$ are totally integrated in the $V$ family, and that their apparent scarcity was an artifact of the conventional nomenclature.

iii) The physical nature and general properties of a mode can be im-

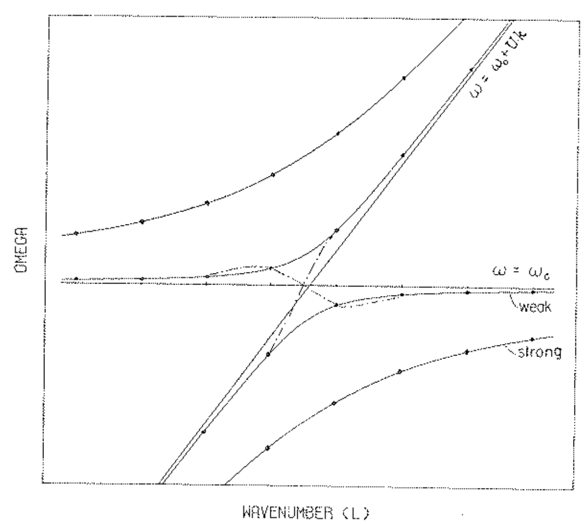


Fig. 10. A simplified model of coupling between two families of modes. Units are arbitrary. Straight lines show the unperturbed dispersion curves. The two hyperbolas represent hybridized dispersion curves for both weak and strong coupling. Dots identify individual modes for a model allowing only for discrete values of the wavenumber. Weak coupling is barely noticed, resulting in slightly irregular pseudo-overtone branches (dash-dot lines); strong coupling extends over several angular orders and generates hybridized branches of modes.

mediately read from its name in the new proposed nomenclature. They will not depend on the model used for their computation (except, obviously, in the case of $H$ modes). In the example chosen in section 2.1, (${}_{26}S_1$ and ${}_{27}S_1$), one of the modes will remain a $K$ mode, and will always be called ${}_7K_1$, the other one will stay a $V$ mode, always called ${}_7V_1$, regardless of the influence on their relative periods of the Earth model being used in the computation.

iv) Figure 9 shows the variations of some properties of $V$, $C$, $K$, and $R$ modes along pseudo-overtone branches $p=\mathrm{cst}$.

A direct comparison with Fig. 2 shows a considerable improvement in the smoothness of these properties, which clearly permits interpolation along the pseudo-overtone branches.

v) The discussion in section 2.3 has shown on a particular example that $U$ was closely related to $U^*$ along a $p$-branch better than along an $n$-branch. This is indeed a general trend, which can be checked all over the $K$, $C$, $V$ and $R$ families.

## 5. *Conclusion*

We have shown that the differences in the physical properties, including group velocity, $Q$, and excitation functions, of the various spheroidal modes have a theoretical origin in the absence of coupling at low angular order. We have also shown that, apart from Inner Core and Stoneley modes, there exist four families of spheroidal modes, corresponding to three different ranges of coupling:

i) Decoupled modes (both $V$ and $C$ families)
ii) Modes with intermediate coupling ($H$ modes)
iii) Fully coupled modes ($R$ modes).

In an $(l, \omega)$ plane, the first group of modes correspond to a domain studied mostly by core waves; the second one is the domain of mantle body waves; the third one of surface wave theory. However, the application of mode theory (especially through a variational approach) to any of the domains could bring insight into the physical properties of the deep mantle. For this purpose, in order to apply the interpolation scheme developed by KANAMORI and STEWART (1976) to higher modes, it is necessary to have a good understanding of the physical properties of the modes over which the interpolation is made. The proposed new nomenclature should help provide that physical insight.

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