36.

Physics of the Earth and Planetary Interiors, 30 (1982) 1–11
Elsevier Scientific Publishing Company, Amsterdam – Printed in The Netherlands

Mode-wave equivalence and other asymptotic problems in tsunami theory

Emile A. Okal

Department of Geology and Geophysics, Yale University, Box 6666, New Haven, CT 06511 (U.S.A.)

(Received January 12, 1982; revision accepted April 13, 1982)

Okal, E.A., 1982. Mode-wave equivalence and other asymptotic problems in tsunami theory. Phys. Earth Planet. Inter., 30: 1–11.

A study is conducted of the asymptotic behavior of the gravity modes of an incompressible spherical oceanic layer, surrounding a rigid Earth, as its radius goes to infinity. The flat-layered Earth dispersion relation $c = \sqrt{gh}$ for the phase velocity of the tsunami wave is derived, and the existence of only one branch of tsunami modes is proved, a result fundamental for the use of mode theory in marigram synthesis. Studied further are the influence of such parameters as finite incompressibility in the ocean, and finite rigidity of the ocean floor, on the dispersion of tsunami modes, both theoretically and numerically for a number of models. It is concluded that for all physically acceptable models of both the ocean and its floor, tsunami dispersion is not significantly affected by either. This includes in particular the case of a sedimentary layer, which is found to have no effect on tsunami propagation. The maximum (and extremely small) rigidity allowed in the fluid before the tsunami modes disappear is also derived.

1. Introduction

Early inroads into tsunami theory were made using the model of a flat incompressible ocean over a rigid half-space, the tsunamigenic source being described as a sudden movement of a portion of the ocean floor. This model, which uses the long-wavelength limit of the theory of gravity waves in liquids (Ben-Menahem and Singh, 1981, pp. 776–796), correctly predicts observed tsunami dispersion, and successfully explains the very large (and devastating) directivity effects due to the disproportion between tsunami phase velocities and earthquake rupture velocities (Ben-Menahem and Rosenman, 1972). However, in this approach, the amplitude of the tsunami in relation to earthquake size is difficult to assess, since it requires a full history of the strong motion of the ocean floor.

Following Abe's (1979) remark that the amplitude of the tsunami is governed by the long-period seismic moment of the source, a major breakthrough came from Ward's (1980) theory describing tsunamis as a particular branch of the spheroidal modes of an ocean-covered spherical Earth. In particular, Ward was able to directly compute tsunami excitation as a function of earthquake depth and for two fundamental source geometries. There remains to be proved, however, the expected result that the flat-layered solution can be obtained asymptotically from the normal mode solution; in this paper, this result is derived, and the existence of only *one* tsunami branch (a result tacitly assumed for the successful computation of synthetic marigrams) is proved. The influence of real-Earth parameters, such as finite compressibility of seawater and finite rigidity of the ocean floor, on those results are then examined theoretically. Although in all physically meaningful situations, this influence is very small if not negligible, these investigations provide

0031-9201/82/0000–0000/$02.75

physical insight into the nature of the phenomena involved in the coupling of tsunami energy with elastic media.

2. Asymptotic behavior: the flat-layered Earth problem

In this paper, the normal modes of a spherically symmetric Earth of radius a, whose shallowest layer is an ocean of depth h and density ρ, are considered. The notation of Saito (1967) is used, which is also that of Ben-Menahem and Singh (1981). In the ocean, the rigidity μ vanishes, and with it, the shear traction y_4. The horizontal displacement, y_3, although non-zero, becomes a spurious variable, satisfying identically

$$y_3 = (gy_1 - y_2/\rho - y_5)/\omega^2 r \tag{1}$$

and the differential system governing the modal solution becomes

$$\frac{d}{dr}\begin{pmatrix} y_1 \\ y_2 \\ y_5 \\ y_6 \end{pmatrix} = M \begin{pmatrix} y_1 \\ y_2 \\ y_5 \\ y_6 \end{pmatrix}$$

with

$$M = \begin{bmatrix} \frac{L^2 g}{\omega^2 r^2} - \frac{2}{r} & \frac{1}{\lambda} - \frac{L^2}{\rho\omega^2 r^2} & -\frac{L^2}{\omega^2 r^2} & 0 \\ -\omega^2\rho - \frac{4\rho g}{r} + \frac{L^2 g^2 \rho}{\omega^2 r^2} & -\frac{L^2 g}{\omega^2 r^2} & -\frac{L^2 g\rho}{\omega^2 r^2} & -\rho \\ 4\pi G\rho & 0 & 0 & 1 \\ -\frac{4\pi L^2 G\rho g}{\omega^2 r^2} & \frac{4\pi G L^2}{\omega^2 r^2} & \frac{4\pi\rho G L^2}{\omega^2 r^2} + \frac{L^2}{r^2} & -\frac{2}{r} \end{bmatrix} \tag{2}$$

as derived readily from Saito (1967). Here $L^2 = l(l+1)$ and λ is Lamé's constant for the ocean. Under the further assumption of an incompressible ocean, $1/\lambda = 0$ can be put in this system. Substituting the first and third line into the fourth

$$\frac{d^2 y_5}{dr^2} + \frac{2}{r}\frac{d y_5}{dr} - \frac{L^2}{r^2} y_5 = 0 \tag{3}$$

which merely expresses the fact that in the absence of compression in the liquid the gravity potential satisfies Laplace's equation. Similarly, after some calculation it is found that y_1 satisfies

$$\frac{d^2 y_1}{dr^2} + \frac{2}{r}\frac{d y_1}{dr} - (L^2 - 2)\frac{y_1}{r^2} = 0 \tag{4}$$

The solution to a gravity mode of the oceanic layer is therefore of the form

$$y_1 = \mathrm{A} r^{l-1} + \mathrm{B} r^{-l-2}$$
$$y_5 = \mathrm{C} r^{l} + \mathrm{D} r^{-l-1} \tag{5}$$

where A, B, C and D are constants.

These solutions generalize Takeuchi and Saito's (1972) third type of solution (their eq. 100, p. 244) to the case where the medium, not extending to the center of the sphere, can entertain waves increasing with $r \to 0$. The dispersion relation $\omega = \omega(l)$ is obtained from the boundary conditions: in order to go asymptotically to the case of the flat ocean, rigid bottom, tsunami model, impose

$$y_1 = y_5 = 0 \qquad \text{for } r = a - h$$
$$y_2 = y_6 + (l+1) y_5 / r = 0 \qquad \text{for } r = a \tag{6}$$

From the equations in system (2)

$$y_2 = -\frac{\omega^2 r^2}{L^2}\rho\frac{d y_1}{dr} - \left(\frac{2\omega^2 r\rho}{L^2} - \rho g\right) y_1 - y_5 \tag{7}$$

is easily derived. It is then straightforward to substitute (5) and (7) into (6) and equate to zero the determinant of a system yielding A, B, C and D. Under the long wavelength assumption ($h \ll a/l$)

$$\omega^2 = \frac{L^2 g}{a} x + 0(x^2), \qquad \text{where } x = h/a \ll 1 \tag{8}$$

or in more usual form, involving the phase velocity, $c^2 = gh$. The solution is then written as

$$\begin{aligned} y_1 &= Y(1-\xi/x) \\ y_2 &= -Y\rho g\xi/x \\ y_3 &= Y/xL^2 \\ y_5 &= Y4\pi\rho Ga(x-\xi) \end{aligned} \tag{9}$$

where Y is a constant, $\xi = (a-r)/a$, and keeping only terms of first order in x, ξ and $1/l$. These expressions, which will be useful in the next sections, are in agreement with Ward's (1980) results. Note in particular that the horizontal displacement ly_3 is constant throughout the oceanic layer, and much larger than the vertical displacement y_1.

Although the asymptotic equivalence between the two models was to be expected, it had never been given in the literature, and it was important to derive, since it provides the basis for the use of normal mode in tsunami studies. In particular, it is clear that under the above conditions, there exists only one branch of tsunami modes, since: (1) the differential equations satisfied by y_1 and y_5 do not involve frequency; (2) only y_2 depends on ω(see eq. 7) and (3) this dependence is linear in ω^2, making the dispersion determinant itself linear in ω^2, which results in only one positive frequency root for each l. Thus, the existence of only one branch of tsunami modes is *not* an artifact of the asymptotic assumption $a \to \infty$. It is clear, from eqs. 5–7 that it holds (although with a more complex dispersion law) even for an oceanic mass of depth comparable to a (until of course gravitational instability takes place). This is a completely different case from that of standard Rayleigh modes, in which the existence of the overtones stems from the transcendental character of the dispersion equation. In the Rayleigh problem, the disappearance of overtone solutions of a homogeneous Earth as $a \to \infty$ for fixed wavenumber k, is due to their eigenfunction remaining finite at depths of the order of a finite fraction of the Earth's radius, itself going to infinity: such solutions cannot be allowed in the flat-layered Earth problem. On the other hand, in the tsunami problem, and even for finite oceanic depths, there is only one branch of solutions.

Thus, in this section, it is proved that the only modes of an incompressible oceanic layer make up one branch of tsunami modes, which under the condition of the long wavelength limit ($lx \ll 1$) propagate at a phase velocity $c = \sqrt{gh}$. It remains to be proved that this result is not significantly affected by either a small value of the compressibility $1/\lambda$ (such as in seawater) or by finite rigidity of the ocean floor.

3. Effect of seawater compressibility

In this section, the assumption of incompressibility of the oceanic layer is relaxed and the effect it has on the existence and dispersion of the tsunami mode is investigated. Similarly, the fluidity of the ocean is later relaxed and we investigate to what extent a solid of very low rigidity can entertain tsunami modes, which are defined as modes in which the potential energy is mostly gravitational. These problems are investigated both theoretically and numerically, using the case of a 4-km deep oceanic layer of density 1 g cm^{-3}, overlaying a rigid homogeneous Earth (it is approximated as a Poisson solid of 10^7 km s^{-1} P-velocity), with a density of 5.52 g cm^{-3}, providing an adequate value of g in the water layer. An angular order of $l = 200$ is used, corresponding to a period of approximately 1000 s for the flat-layered Earth incompressible model.

3.1. Effect of finite bulk modulus

In this section, the term $1/\lambda$ is reintroduced in system (2), while keeping the fluidity of the medium ($\mu = 0$). The long-wavelength approximation $lx \ll 1$ is also kept. Furthermore, the variable y_2 is replaced by $y_7 = y_2/\lambda$, which represents the dilatation $u_{i,i}$ in the seawater. Equation 2 can be rewritten with the variables y_1, y_7, y_5 and y_6, using the matrix

$$M=\begin{bmatrix} \frac{L^2 g}{\omega^2 r^2}-\frac{2}{r} & 1-\frac{\lambda L^2}{\rho\omega^2 r^2} & -\frac{L^2}{\omega^2 r^2} & 0 \\ -\frac{\omega^2\rho}{\lambda}-\frac{4\rho g}{\lambda r}+\frac{L^2 g^2\rho}{\lambda\omega^2 r^2} & -\frac{L^2 g}{\omega^2 r^2} & -\frac{L^2 g\rho}{\lambda\omega^2 r^2} & -\frac{\rho}{\lambda} \\ 4\pi G\rho & 0 & 0 & 1 \\ -\frac{4\pi L^2 G\rho g}{\omega^2 r^2} & \frac{4\pi G L^2\lambda}{\omega^2 r^2} & \frac{4\pi\rho G L^2}{\omega^2 r^2}+\frac{L^2}{r^2} & -\frac{2}{r} \end{bmatrix} \tag{10}$$

The resulting differential system can be considered as expressing the coupling of the following oscillators:

A tsunami mode, corresponding to $1/\lambda=0$ and satisfying eq. 3. In this mode $y_7=0$ but y_2 remains finite.

Seismic modes, corresponding to $G=0$, $g=0$ and satisfying

$$\frac{d^2 y_7}{dr^2}+\frac{2}{r}\frac{dy_7}{dr}-\left[\frac{L^2}{r^2}-\frac{\omega^2\rho}{\lambda}\right]y_7=0 \tag{11}$$

The latter, in which $y_5=y_6=0$, represent, of course, seismic energy trapped in the water layer, with only little dependence on the horizontal coordinate ($lx \ll 1$). In the absence of coupling, their eigenfrequencies are governed by the depth to ocean floor, and are of the order of $\omega=p\alpha/ax$ ($p \gtrsim 1$) for the fundamental and its overtones (note that this result is independent of l). For all such modes, one has a phase velocity much larger than the P-wave velocity α, and eq. 11 degenerates to $d^2y_7/dr^2+(\omega^2/\alpha^2)y_7=0$, all other terms becoming negligible.

In the presence of coupling, eqs. 3 and 11 are replaced by

$$\frac{dy_5}{dr^2}+\frac{2}{r}\frac{dy_5}{dr}-\frac{L^2}{r^2}y_5=4\pi G\rho y_7 \tag{12}$$

$$\frac{d^2y_7}{dr^2}+\frac{2}{r}\frac{dy_7}{dr}+\left\{\frac{\omega^2\rho}{\lambda}-\frac{L^2}{r^2}\right.$$
$$\left.\left[1-\frac{4\rho gr}{\lambda L^2}+\frac{g^2\rho}{\lambda\omega^2}+\frac{6g}{\omega^2 r}-\frac{8\pi G\rho}{\omega^2}\right]\right\}y_7=\frac{g\rho}{\lambda r}\left[\frac{6L^2}{\omega^2 r^2}\right.$$
$$\left.-\frac{8\pi G\rho r L^2}{g\omega^2 r^2}\right]y_5+\frac{\rho}{\lambda r}\left[\frac{6g}{4\pi G\rho r}\left(2-\frac{gL^2}{r\omega^2}\right)\right.$$
$$\left.-4+\frac{2L^2 g}{r\omega^2}\right]\left[\frac{dy_5}{dr}-y_6\right] \tag{13}$$

the boundary conditions remaining unchanged.

Because r varies very little (about 1/1000) in the oceanic layer, the effect of the new y_7 terms in the l.h.s. of eq. 13 can be viewed as only slightly modifying the apparent value of l in eq. 11. The eigensolutions to the l.h.s. of eq. 13, governed by the unchanged boundary conditions, and independent of l, are practically unchanged, and therefore remain outside the range of tsunamis ($c \gg \alpha$; no significant gravitational energy).

When dealing with the r.h.s. of eqs. 12 and 13, it is possible to compute a dimensionless estimate of the intensity of coupling from an examination of the orders of magnitude of the terms involved. The coupling (say ϵ) can be described by the square root of the product of the dominant r.h.s. coefficients, divided by typical eigenvalues of the l.h.s. operators. In the present case, it is found that the dominant r.h.s. term in eq. 13 is the last term, of the order of

$$(2L^2g/r\omega^2)(1-3g/4\pi G\rho r)y_5/\lambda a^2 x$$
$$\simeq (9L^2 g/r\omega^2)\rho y_5/\lambda a^2 x$$

In the frequency range of tsunamis, typical eigenvalues of the l.h.s. operators are $1/a^2x^2$, leading to

$$\epsilon^2=36\pi Gg\rho^2L^2ax^3/\lambda\omega^2; \qquad \epsilon \lesssim 3\% \tag{14}$$

for seawater and for all $\omega>0.03$ rad s^{-1} and all $l>100$. Since the anticipated coupling is so small, one is justified in using linear perturbation theory to predict that the tsunami mode will see its eigenfunction unchanged at first order in $1/\lambda$ and its eigenfrequency shifted by an amount readily computed from the variational formula (Takeuchi and Saito, 1972, eq. 201)

$$\frac{\delta\omega}{\omega}=-\frac{1}{\lambda}\frac{\int(ry_2)^2\,dr}{2\omega^2\int\rho(y_1^2+L^2y_3^2)r^2\,dr} \tag{15}$$

where the integrals can be conveniently limited to the oceanic layer. Using formulae 9

$$\frac{\delta\omega}{\omega} = -\frac{c^2}{6\alpha^2} \qquad (16)$$

Similarly, the seismic modes (fundamental and overtones) would take up a little gravitational energy, and see their frequencies slightly affected. The most important result is that as long as linear perturbation theory can be applied, only *one* tsunami branch occurs, i.e. one tsunami mode for each l.

If coupling is strong, in particular when α becomes comparable to the unperturbed c, it ceases to be linear, and the behavior of the solution cannot be predicted simply. For all practical applications, tsunamis are considered only in seawater, where the P-wave velocity is fixed to about 1.5 km s^{-1}, so that strong coupling would occur only if the unperturbed tsunami phase velocity reached that value. That would require a depth of the order of 250 km, which is totally unrealistic.

These theoretical results are checked in Table I against tsunami eigenfrequencies for $l = 200$, computed using a spherical model with a 4-km deep oceanic layer over a rigid solid Earth. In order to investigate the asymptotic influence of small $1/\lambda$ several hypothetic values of α in the water are considered, from 0.1 to 100 km s^{-1}. As long as α remains larger than the unperturbed phase velocity (about 200 m s^{-1}), linear perturbation theory gives excellent results, as shown by columns 2, 4 and 6 of Table I. In particular, for seawater (1.5 km s^{-1}), the frequency shift is only about 0.3%, a figure equal to one-half the proportion of elastic energy in the system. The linearity breaks down around $\alpha = 200$ m s^{-1}, with the potential energy becoming strongly elastic, and the tsunami mode disappearing.

In conclusion, all realistic models of seawater and oceanic depths predict one and only one tsunami branch, obeying the dispersion relation

$$c^2 = gh\left(1 - gh/3\alpha^2\right)$$

3.2. *Effect of small rigidity*

The purpose of this section is to investigate to what extent rigidity in the ocean would prevent propagation of a tsunami mode. Although a model involving rigidity in the ocean is clearly unrealistic, it will help in attacking the problem of the in-

TABLE I

Influence of ocean compressibility on tsunami dispersion (rigid ocean floor; depth of ocean column = 4 km)

P-wave velocity		Period of tsunami mode for $l = 200$		Potential energy of mode	
km s^{-1}	$\alpha^2/3c^2$	Seconds	$-2\delta\omega/\omega$ *	Gravitational	Elastic
100	1.30E-6	1010.68	—	1.0000	1.28E-6
30	1.44E-5	1010.69	1.E-5	1.0000	1.43E-5
10	1.30E-4	1010.75	1.39E-4	0.9999	1.28E-4
5	5.20E-4	1010.94	5.15E-4	0.9995	5.13E-4
3	1.45E-3	1011.40	1.43E-3	0.9986	1.43E-3
2	3.25E-3	1012.30	3.21E-3	0.9968	3.21E-3
1.5 **	5.78E-3	1013.57	5.72E-3	0.9943	5.72E-3
1.0	1.30E-2	1017.21	1.29E-2	0.9871	1.29E-2
0.7	2.65E-2	1024.11	2.66E-2	0.9734	2.67E-2
0.5	5.20E-2	1037.38	5.28E-2	0.9470	5.30E-2
0.2	0.33	1205.28	0.385	0.6210	0.379
0.1	No tsunami solution				

* Reference "incompressible" case is taken as 1010.68 s.

** Seawater.

fluence of sedimentary layers and other non-rigid ocean bottom properties on tsunami propagation, by giving an insight into the ability of the tsunami mode to penetrate such layers.

Once a small $\mu(\mu \ll \lambda)$ is inserted into the problem, we must revert to the full 6×6 system described, for example, by Ben-Menahem and Singh (1981, eqs. 6.151–6.156, p. 373). An order-of-magnitude study of the various new terms suggests that the perturbation they introduce is of the order of β^2/c^2, comparable to the ratio of the shear elastic potential energy to the total energy of the system. However, the variational formalism used in the previous section cannot be used here, because the boundary conditions have been changed drastically by relaxing the fluidity of the ocean. Most significantly, y_3 must now be continuous (in practice, zero) at the bottom of the ocean. Since the structure of the tsunami wave rested on very large horizontal flow in the oceanic layer (see eq. 9), it is anticipated that the breakdown of the tsunami mode will occur for velocities β much smaller than in the case of the P-wave velocities of section 3.1. This is clearly confirmed by the data in Table II which present the results of dispersion calculations ($l = 200$) for various values of a small rigidity, added to the seawater model of Table I ($\alpha = 1.5$ km s^{-1}). Even for an extremely small rigidity ($\beta = 0.03$ km s^{-1}, Poisson ratio $\nu = 0.4998$), the tsunami mode is destroyed. The systematic investigation of the influence of even smaller values of β shows that tsunami modes can exist for shear velocities of the order of 8–10 m s^{-1}, although these solutions are strongly perturbed, and carry only about 60% gravitational energy. As β is diminished, the tsunami period becomes close to the unperturbed solution. However, when $\beta \leq 0.006$ km s^{-1}, strong coupling takes place, since the seismic shear modes of the ocean column (or their overtones) take a frequency comparable to 1000 s. This results in the development of two strongly hybridized solutions, whose frequencies vary irregularly with β. The physical nature of this phenomenon is similar to coupling

TABLE II

Influence of rigidity in ocean on tsunami modes (rigid ocean floor; depth of ocean column = 4 km)

S-wave velocity		Period of modes for $l = 200$ (s)	Potential energy of mode		Type of solution *
m s^{-1}	Poisson ratio		Gravitational	Elastic	
Tsunami solution for the fluid case					
0	0.5	1013.57	0.9943	0.0057	T
Tsunami solution strongly coupled and hybridized to seismic modes					
1		1008.93	0.6395	0.3605	T
		1080.90	0.1661	0.8339	S
3		964.10	0.5316	0.4684	T
		1148.92	0.3093	0.6907	S
5		923.99	0.4961	0.5039	H
		1235.59	0.3415	0.6585	S
6		1119.02	0.5475	0.4525	T
		825.46	0.2687	0.7314	S
Tsunami solution					
8		1006.09	0.6547	0.3453	T
10	0.5000	929.63	0.6142	0.3858	T
No tsunami solution					
20	0.4999	647.71	0.3222	0.6778	S
30	0.4998	477.30	0.1767	0.8233	S

* T, tsunami mode (predominantly gravitational energy); S, seismic mode (predominantly elastic energy); H, fully hybridized mode.

between spheroidal modes belonging to different families in the high-phase velocity part of the dispersion plane, as studied by Okal (1978).

The fact that such a small rigidity is enough to inhibit development of a tsunami mode is puzzling and warrants some physical discussion. A tsunami mode consists of energy oscillating between kinetic and gravitational potential forms. This is possible only if the displacement field involves no substantial elastic potential energy. In an oceanic column, vertical displacement alone with not do, since it has to vanish (or be extremely small) on the ocean floor, and thus involve a strong vertical strain e_{zz}. Hence the role played by the prominent horizontal component of the displacement which, in the regular tsunami wave, reaches to the bottom of the ocean with its amplitude practically undisturbed (see eq. 9). This horizontal flow provides the reservoir of kinetic energy (necessary for the oscillation with the gravitational potential energy), to which the vertical displacement contributes very little. As soon as rigidity is allowed in the ocean, a welded boundary condition is imposed on the sea floor, and since the tsunami wave can hardly penetrate the solid Earth, this in practice constrains the horizontal flow to vanish on the ocean floor, and leads to a substantial shear strain component e_{xz} (the contribution to e_{xz} of the vertical motion is negligible in the long-wavelength approximation $lx \ll 1$). This necessary shear strain forces a tapping of the kinetic energy for shear potential energy, and the tsunami mode can only subsist if the ratio of the shear potential energy to the kinetic energy is kept small, and most of the kinetic energy remains available for gravitational potential energy. Assuming a value of e_{xz} independent of z (actually, a less favorable energy pattern will be required since e_{xz} must vanish at the surface of the ocean), this ratio is found to be

$$\tfrac{1}{4}\int_0^x \mu(\partial y_3/a\partial\xi)^2 a\,\mathrm{d}\xi \Big/ \tfrac{1}{2}\int_0^x \rho y_3^2\omega^2 a\,\mathrm{d}\xi = \tfrac{3}{2}\mu/\rho\omega^2 a^2 x^2 \qquad (17)$$

which is of the order of $\frac{3}{2}(\beta/clx)^2$.

Keeping a significant tsunami in the water layer (say 75% of the potential energy gravitational) clearly requires shear velocities in the range of $10\ \mathrm{m\ s^{-1}}$.

As models involving such low shear velocities are totally unrealistic, this section is simply concluded by asserting that tsunami modes cannot exist in any solid worthy of this name, even with a Poisson ratio as high as 0.499.

4. Influence of finite rigidity of the ocean floor

In this section, we revert to a fluid ocean (and the theoretical investigations to an incompressible one), and the influence on tsunami dispersion of finite rigidity of the ocean floor is studied. In particular, the possibility that transitional sedimentary layers might play the role of impedance adaptators is explored.

For this purpose, refer back to system (2), and replace the boundary condition $y_1 = 0$ for $r = a(1-x)$ with $y_2 = y_1 Z$, where Z is the mechanical impedance of the substratum at frequency ω. Also, since it is known from section 3 that a tsunami wave cannot penetrate a solid, the perturbation in the gravitational potential in the ocean floor is expected to be merely $g y_1$. Therefore replace the boundary condition $y_5 = 0$ by $y_5 = g y_1$.

Treat the impedance function $Z(\omega)$ as a known function, which can be computed, for example, through the use of a Haskell propagator in the solid ocean floor. The solution to the tsunami dispersion problem can again be sought in terms of the constants A, B, C and D of eq. 5; it now requires vanishing of the determinant

$$\Delta = \begin{vmatrix} -\dfrac{\omega^2}{L^2}\rho(l+1)(a-h)^l + (\rho g - Z)(a-h)^{l-1} & \dfrac{\omega^2}{L^2}\rho l(a-h)^{-l-1} + (\rho g - Z)(a-h)^{-l-2} & -\rho(a-h)^l & -\rho(a-h)^{-l} \\ -\dfrac{\omega^2\rho}{L^2}(l+1)a^l + \rho g a^{l-1} & \dfrac{\omega^2\rho}{L^2} l a^{-l-1} + \rho g a^{-l-2} & -\rho a^l & -\rho a^{-l} \\ -g(a-h)^{l-1} & -g(a-h)^{-l-2} & (a-h)^l & (a-h)^{-l} \\ -4\pi G\rho a^{l-1} & -4\pi G\rho a^{-l-2} & (2l+1)a^{l-1} & a^{-l-1} \end{vmatrix} \qquad (18)$$

Neglecting terms of second order in $x = 1 - r/a$, the characteristic equation can be reduced to

$$\frac{a^2 x}{L^2}\omega^4 - \frac{ag\Upsilon}{L^2}\omega^2 + g^2 x\Upsilon = 0 \qquad (19)$$

where the non-dimensional parameter

$$\Upsilon = Z/\rho g \qquad (20)$$

is introduced, which is the ratio of the impedances of the solid floor and of the incompressible oceanic layer, the pressure in the ocean being due to the gravitational potential. The case of a rigid ocean floor corresponds to $\Upsilon = \infty$. For Υ not infinitesimally small, and $x \ll 1$, eq. 19 has two roots

$$\omega^2 = gxL^2/a \quad \text{and} \quad \omega^2 = g\Upsilon/ax \qquad (21)$$

The first solution is the tsunami mode, whose dispersion is found to be independent of Υ, and thus of the material properties of the ocean floor, at first order in x. As for the second solution, it must be considered an implicit equation for ω, involving the known function $Z(\omega)$. Although in principle $Z(\omega)$ should be computed in the spherical Earth model, it can be approximated with the model of a homogeneous half-space substratum. In the case of a Poisson-solid of rigidity μ, it is easily shown that

$$Z(\omega) = \mu k \frac{(2-\kappa^2)^2 - 4\sqrt{(1-\kappa^2)(1-\kappa^2/3)}}{\kappa^2\sqrt{1-\kappa^2/3}} \qquad (22)$$

where k is the horizontal wavenumber and $\kappa = \omega/\beta k$. Since only solutions decaying with depth in the substratum are allowed, one must have $\kappa < 1$. The second solution in (21) can be rewritten

$$\kappa^2 lx = Z(\omega)/\mu k \qquad (23)$$

and since κ is bounded and $lx \ll 1$, it requires $Z \simeq 0$, which is, of course, the Rayleigh mode: $\kappa = 0.919$.

In the case of layering, or of a solid with a different Poisson ratio, the reasoning would be exactly similar, and the second solution would correspond to the Rayleigh modes, including any overtones.

Thus, it is shown that a finite ocean floor rigidity does not, at first order in x, perturb the dispersion of the tsunami solution, and keeps the characteristics of the Rayleigh modes. These results are confirmed in Table III, which compiles data obtained for various solid substrata. However, if the impedance of the substratum is decreased to give Υ small values comparable to 1, additional terms in eq. 21, of order l^2x^2/Υ, may become detectable, if not fully significant. In particular, for a reasonable crustal velocity value ($\alpha = 6$ km s^{-1}), the tsunami frequency has shifted about 2 s or 0.2%. Once again, only *one* tsunami branch is present, and the only assumption on the ocean floor is that it cannot be penetrated by a tsunami-type mode, in other words, and according to section 3.2, that it is solid. This can be best illustrated if these results are now used to investi-

TABLE III

Influence of ocean floor rigidity on tsunami dispersion (depth of seawater column = 4 km; ocean floor is a Poisson solid)

P-wave velocity (km s^{-1})	Υ at 1014 s	Period of tsunami mode for $l = 200$		Potential energy of mode	
		Seconds	$-\delta\omega/\omega$ *	Gravitational	Elastic
1.0E+7	1.E+14	1013.57	—	09943	5.72E-3
100	15 435	1013.57	0	0.9943	5.72E-3
30	1 279	1014.16	5.82E-4	0.9940	5.99E-3
10	142.0	1014.71	1.13E-3	0.9919	8.09E-3
5	35.41	1016.59	2.98E-3	0.9848	1.52E-2
3	12.66	1021.09	7.42E-3	0.9674	3.26E-2
1.5	3.07	1044.34	3.04E-2	0.8680	0.1320

* Reference "rigid floor" case is taken as 1013.57 s.

TABLE IV

Influence of Poisson-solid sedimentary layer on tsunami dispersion (depth of ocean column = 4 km) *

P-wave velocity in sediments		Period of tsunami mode for $l = 200$ (s)	Potential energy of mode	
km s^{-1}	Poisson ratio		Gravitational	Elastic
6	0.25	1015.82	0.9877	1.23E-2
4	0.25	1015.86	0.9876	1.24E-2
2.5	0.25	1015.95	0.9874	1.26E-2
2.0	0.25	1016.03	0.9872	1.28E-2
1.5	0.25	1016.21	0.9869	1.31E-2
1.0	0.25	1016.71	0.9859	1.41E-2

* Sediments overlay Poisson-solid crust with 6 km s^{-1} P-wave velocity. Sediments are taken as Poisson solid of variable P-wave velocity.

TABLE V

Influence of rigidity of sedimentary layer on tsunami dispersion (depth of ocean column = 4 km) *

Reference values in the absence of sediments

Depth of ocean (km)	Period of tsunami mode for $l = 200$ (s)	Potential energy of mode	
		Gravitational	Elastic
4	1015.82	0.9877	0.130E-1
5	990.00	0.9841	0.159E-1

Models including 1-km sedimentary layer

S-wave velocity in sediments		Period of tsunami mode for $l = 200$ (s)	Potential energy of mode	
km s^{-1}	Poisson ratio		Gravitational	Elastic
Tsunami regime in 4-km deep ocean				
0.866	0.25	1016.21	0.9869	1.31E-2
0.4	0.462	1016.19	0.9870	1.30E-2
0.2	0.491	1016.19	0.9870	1.30E-2
0.1	0.498	1016.21	0.9870	1.30E-2
0.01	0.500	1019.38	0.9801	1.99E-2
0.005	0.500	1037.76	0.9123	8.77E-2
Hybridized regime				
0.004	0.500	1073.55	0.7217	0.2783
		879.25	0.4010	0.5990
0.0035	0.500	1130.35	0.4955	0.5045
		935.09	0.6602	0.3398
Tsunami regime in 5-km deep ocean				
0.003	0.500	966.62	0.8535	0.1465
0.002	0.500	990.33	0.9534	4.66E-2
0.001	0.500	990.21	0.9470	5.30E-2

* Sediments overlay Poisson-solid crust with 6 km s^{-1} P-wave velocity. P-wave velocity of sediments kept at 1.5 km s^{-1}.

gate the effect, if any, of a sedimentary layer on the dispersion of tsunami modes. As long as the rigidity of the sediments is sufficient to prevent the development of a tsunami mode inside them (that this is the case for $\beta \geqslant 0.01$ km s^{-1} has been seen), the sediments and the substratum make up the solid in the above discussion, and although the expression for $Z(\omega)$ is more complex, the same reasoning holds. Therefore, for all physically acceptable models, sediments are expected to play no significant role in the dispersion of tsunami modes. In particular, they cannot have the coupling effect of impedance adaptators, identified for short-period Rayleigh waves by Okal and Talandier (1981). Tables IV and V are a check of these results against real computations.

In Table IV, the sediments are constrained to being Poisson solids, and their P-wave velocity varies from that of the crust (6 km s^{-1}) to that of the oceanic layer (1.5 km s^{-1}). The total shift of frequency for tsunami modes computed in these models is less than 0.1%. On the other hand, in Table V, we examine the influence on the dispersion of a variation in the Poisson ratio of the sediments, having fixed their P-wave velocity to 1.5 km s^{-1}. Three regimes are clearly identified: for $\beta > 0.01$ km s^{-1}, and as predicted in section 3.2, the tsunami cannot penetrate the sedimentary layer, and the solution essentially remains that for the 4-km deep ocean. For values of $\beta \leqslant 0.003$ km s^{-1}, it is possible to develop a tsunami wave in the sedimentary material, and therefore the tsunami mode "jumps" the sediments and becomes essentially the solution for a 5-km deep ocean. (No coupling with seismic modes is involved since the thickness of the shear layer has become too small.) In between, and for a limited range of values of β, the two modes are coupled with their eigenfunctions strongly hybridized.

Again, and for all physically acceptable sedimentary models, no significant effects on the dispersion can take place.

5. Conclusion

The main results obtained in this paper can be listed as follows:

(1) The flat-layered ocean tsunami solution and its dispersion as an asymptotic limit of the normal modes of a spherical oceanic shell are derived.

(2) It is proved that only one branch of tsunami modes exists, the absence of overtones not being an artifact of the Earth-flattening approximation.

(3) The frequency shift due to finite incompressibility in the oceanic layer $\delta\omega/\omega = -c^2 6/\alpha^2$ is derived.

(4) It is proved that even a minute rigidity is sufficient to inhibit the development of a tsunami-type gravity mode in an oceanic layer.

(5) It is shown that finite rigidity of the ocean floor has no effect on tsunami dispersion. In particular, the dispersion is not affected by sedimentary layers, however soft.

Although the mode-wave equivalence was to be expected, and some of the quantitative bounds derived in the other results are extremely small, and do not put any constraints on real-Earth situations, at least one fundamental point has been proved in the paper: that under all physically acceptable conditions for both the seawater column and the ocean floor, only one branch of tsunami modes can exist. This justifies the use of normal mode synthetics in the reconstruction of marigrams.

There remains, however, the problem of the possible influence of a sedimentary layer on the *excitation* of a tsunami mode, whether or not the seismic source penetrates it. Because a softer layer may provide for a different (but still highly inhibited) penetration of the eigenfunction inside the solid Earth, this influence may be relatively more important than that on the eigenfrequency. Such a mechanism has been suggested by Fukao (1979), to explain the generation of a stronger-than-expected tsunami following the 1975 Kuriles earthquake, in an area where a substantial sedimentary wedge is present at the collision zone between the two plates. This investigation will be the subject of a separate paper.

Acknowledgments

The calculation of tsunami modes was performed using a computer program originally written by Ralphe Wiggins, and on which Robert Geller provided useful comments. This research was supported by the National Science Foundation, under Grant EAR 81-06106, and the National Oceanic and Atmospheric Administration, under Grant NA80-RAD-00014.

References

Abe, K., 1979. Size of great earthquakes of 1837–1974 inferred from tsunami data. J. Geophys. Res., 84: 1561–1568.

Ben-Menahem, A. and Rosenman, M., 1972. Amplitude patterns of tsunami waves from submarine earthquakes. J. Geophys. Res., 77: 3097–3128.

Ben-Menahem, A. and Singh, S.J., 1981. Seismic Waves and Sources. Springer-Verlag, New York, NY, 1108 pp.

Fukao, Y., 1979. Tsunami earthquakes and subduction processes near deep-sea trenches. J. Geophys. Res., 84: 2303–2314.

Okal, E.A., 1978. A physical classification of the Earth's spheroidal modes. J. Phys. Earth, 26: 75–103.

Okal, E.A. and Talandier, J., 1981. Dispersion of one-second Rayleigh modes through oceanic sediments following shallow earthquakes in the Southcentral Pacific. In: W. Kupermann and F. Jensen (Editors), Bottom Interacting Ocean Acoustics, NATO Series IV: 5. Plenum, New York, NY, pp. 345–358.

Saito, M., 1967. Excitation of free oscillations and surface waves by a point-source in a vertically heterogeneous Earth. J. Geophys. Res., 72: 3689–3699.

Takeuchi, H. and Saito, M., 1972. Seismic surface waves. Methods Comput. Phys., 12: 217–295.

Ward, S.N., 1980. Relationships of tsunami generation and an earthquake source. J. Phys. Earth, 28: 441–474.