Geology 202 Problem Set 5
The Early Solar System

Due in one week Name ________________________________

1. The gravitational energy binding an object of mass $m$ to a planet of mass $M$ and radius $r$ is $U = GMm/r$. Since this is also the energy that would be released by bringing the object to earth from infinity, this relation is used to calculate the energy in accreting planets.

a) Assuming all this energy is converted to kinetic energy, find the velocity that an object would impact with on earth today.

$$U = \frac{GMm}{r} = \frac{1}{2} mv^2$$

The equation can be rearranged to solve for v:

$$v = \left(\frac{2GM}{r}\right)^{1/2}$$

where $G = 6.67x10^{-11} Nm^2/kg^2$, $r = 6.371x10^6 m$, and $M = 5.95x10^{24} kg$. Thus v = 11161.76 m/s = 11.16176 km/s.

b) What would the impact velocity be during the early stages of earth formation? Assume the earth had radius 100 km and the present average density.

As shown in part a,

$$v = \left(\frac{2GM}{r}\right)^{1/2}$$

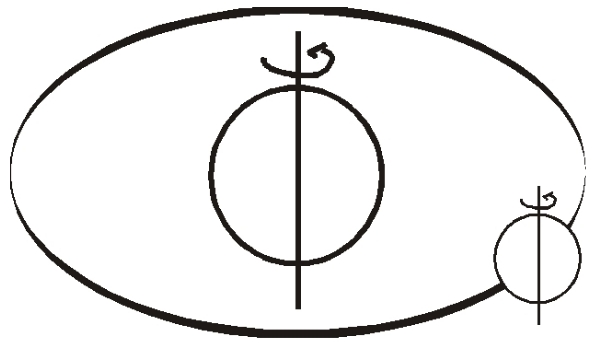
Since the radius is only 100 km, the mass is not the same as in part a. Using the average density as $5500 kg/m^3$ and the volume of a sphere, the mass is:

$$M = \left(5500 kg/m^3\right) x \left(4/3\pi\left(100x10^3 m\right)^3\right) = 2.3x10^{19} kg$$

$$v = \left(\frac{2\left(6.67x10^{-11} Nm^2/kg^2\right)\left(2.3x10^{19} kg\right)}{100x10^3 m}\right)^{1/2} = 175.2 m/s = 0.1752 km/s$$

2. Angular momentum in the solar system.

Angular momentum is the momentum associated with rotating bodies. As all bodies in motion have a *linear* momentum, so do all rotating bodies have an associated *angular* momentum. Like linear momentum, angular momentum is conserved, so, in the absence of external torques (or equivalently in linear motion: forces), the angular momentum remains constant with time. In considering the solar system there are two forms of circular motion to consider



a) Rotation of a body about a spin-axis
For a rigid body rotating about a fixed axis, the angular momentum is,

$$L_{rot} = I\omega_{rot},$$

where I is the moment of inertia of the body and $\omega_{rot}$ is the angular velocity in radians/sec. Assume that the Sun and planets are rigid bodies (A statement that would make any true celestial mechanic laugh), and recall that the moment of inertia factor for a body of mass $M$ and radius $a$ is equal to;

$$k = I/Ma^2$$

The angular velocity can be calculated by dividing the number of radians ($2\pi$ radians = 360°) traversed in one rotation by the rotational period, the time it takes for the planet to rotate on its axis. On Earth, for example, this rotation requires 1 day. Using the mass, radius, period and moment of inertia factors given below, calculate the angular velocity, moment of inertia and angular momentum for the bodies listed, **using the units listed in the table!**

Here is the completed table. For an explaination, see the equations below the table.

Angular Momentum associated with rotation about an axis.

| Body | Rotation Period days | Equatorial Radius km | Mass $10^{24}$ kg | Moment of Inertia Factor | $\omega_{rot}$ rad/sec | Moment of Inertia kg-$m^2$ | $L_{rot}$ kg-$m^2$/sec |
|---|---|---|---|---|---|---|---|
| Sun | 27 | $6.96 \times 10^5$ | $2 \times 10^6$ | 0.006 | $2.69 \times 10^{-6}$ | $5.81 \times 10^{45}$ | $1.57 \times 10^{40}$ |
| Earth | 1 | 6,378 | 5.97 | 0.331 | $7.27 \times 10^{-5}$ | $8.04 \times 10^{37}$ | $5.85 \times 10^{33}$ |
| Jupiter | 0.40 | 71,900 | 1900 | 0.25 | 0.000182 | $2.46 \times 10^{42}$ | $4.46 \times 10^{38}$ |
| Saturn | 0.43 | 60,200 | 570 | 0.22 | 0.000169 | $4.55 \times 10^{41}$ | $7.69 \times 10^{37}$ |

To make the equations shorter, note that there are 86400 seconds in one day.

Sun:

$$\omega_{rot} = \frac{2\pi}{27 days x 86400 seconds/day} = 2.69 x 10^{-6} radians/seconds$$

$$I = 0.006 x \left(2 x 10^6 x 10^{24} kg\right) x \left(6.96 x 10^5 km x 1000 m/km\right)^2 = 5.82 x 10^{45} kgm^2$$

$$L_{rot} = I\omega_{rot} = 1.57 x 10^{40} kgm^2/s$$

Earth:

$$\omega_{rot} = \frac{2\pi}{1 days x 86400 seconds/day} = 7.27 x 10^{-5} radians/seconds$$

$$I = 0.331 x \left(5.97 x 10^{24} kg\right) x (6378 km x 1000 m/km)^2 = 8.04 x 10^{37} kgm^2$$

$$L_{rot} = I\omega_{rot} = 5.85 x 10^{33} kgm^2/s$$

Jupiter:

$$\omega_{rot} = \frac{2\pi}{0.40 days x 86400 seconds/day} = 0.000182 radians/seconds$$

$$I = 0.25 x \left(71900 x 10^{24} kg\right) x (1900 km x 1000 m/km)^2 = 2.46 x 10^{42} kgm^2$$

$$L_{rot} = I\omega_{rot} = 4.46 x 10^{38} kgm^2/s$$

Saturn:

$$\omega_{rot} = \frac{2\pi}{0.43 days x 86400 seconds/day} = 0.000169 radians/seconds$$

$$I = 0.22 x \left(570 x 10^{24} kg\right) x (60200 km x 1000 m/km)^2 = 4.55 x 10^{41} kgm^2$$

$$L_{rot} = I\omega_{rot} = 7.69 x 10^{37} kgm^2/s$$

To see where most of the rotational angular momentum resides, compare the angular momentum ratio of the Earth/Sun and Jupiter/Sun (Hint: setting up ratios may be the easiest way to do this).

$$Earth/Sun = \frac{5.85 x 10^{33} kgm^2/s}{1.57 x 10^{40} kgm^2/s} = 3.73 x 10^{-7}$$

$$Jupiter/Sun = \frac{4.46 x 10^{38} kgm^2/s}{1.57 x 10^{40} kgm^2/s} = 0.0285$$

The Jupiter/Sun ratio is larger than the Earth/Sun ratio showing that more rotational angular moemomentum resides in Jupiter than in Earth.

b) Orbital motion of a body about the center of mass.
Assume that the planets represent point masses which revolve about the center of mass of the solar system (which to a very good approximation is the center of the Sun). Also assume that the planets orbit in a plane (known as the *ecliptic*) perpendicular to the spin axis of the sun. You can then approximate the angular momentum associated with the revolution of the planets about the sun by the expression,

$$L_{orb} = mr^2\omega_{orb}$$

where $m$ is the mass of the planet,$r$ is the orbital radius and $\omega_{orb}$ is the angular velocity of the planet in its orbit (which again is $2\pi$ radians divided by the time required for a complete orbit). Using the values for orbital period and radius given below, and the masses given in part (a), calculate the angular velocity and angular momentum associated with the orbits of the planets. Enter these in the table below (again in the **units listed**).

Here is the completed table. For an explanation, see the equations below the table.

| Angular Momentum associated with revolution about a point. | | | | |
|---|---|---|---|---|
| Body | Revolution Period yrs | Mean Orbital Radius $10^6$ km | $\omega_{orb}$ rad/sec | $L_{orb}$ kg-$m^2$/s |
| Earth | 1 | 150 | $1.99\ x\ 10^{-7}$ | $2.68\ x\ 10^{40}$ |
| Jupiter | 11.9 | 778 | $1.67\ x\ 10^{-8}$ | $1.93\ x\ 10^{43}$ |
| Saturn | 29.5 | 1427 | $6.75\ x\ 10^{-9}$ | $7.84\ x\ 10^{42}$ |

To make the equations shorter, note that there are 31536000 seconds in one year.

Earth:

$$\omega_{orb} = \frac{2\pi}{1 years x 31536000 seconds/years} = 1.99x10^{-7} radians/second$$

$$L_{orb} = \left(5.97x10^{24}kg\right)x\left(150x10^6x10^3m\right)^2x\left(1.99x10^{-7}radians/second\right) = 2.68x10^{40}kgm^2/s$$

Jupiter:

$$\omega_{orb} = \frac{2\pi}{11.9 years x 31536000 seconds/years} = 1.67x10^{-8} radians/second$$

$$L_{orb} = \left(1900x10^{24}kg\right)x\left(778x10^6x10^3m\right)^2x\left(1.67x10^{-8}radians/second\right) = 1.93x10^{43}kgm^2/s$$

Saturn:

$$\omega_{orb} = \frac{2\pi}{29.5 years x 31536000 seconds/years} = 6.75x10^{-9} radians/second$$

$$L_{orb} = \left(570x10^{24}kg\right)x\left(1427x10^{6}x10^{3}m\right)^{2}x\left(6.75x10^{-9}radians/second\right) = 7.84x10^{42}kgm^{2}/2$$

Where does most of the orbital angular momentum reside?

Most of the orbital angular momental resides in Jupiter

c) Sum up all the angular momentum (rotational and orbital) in the solar system (just consider the bodies for which you have calculated the angular momenta). Compare this to the total angular momentum individually associated with each: the Sun, Earth, Jupiter and Saturn. Where does the majority of the angular momentum reside?

The total angular momentum is $2.71x10^{43}kgm^{2}/s$.

The total angular momentum of the Sun is $1.57x10^{40}kgm^{2}/s$, which is 0.058% of the total angular momentum for the solar system. The total angular momentum of the Earth is $2.68x10^{40}kgm^{2}/s$, which is 0.099% of the total angular momentum for the solar system. The total angular momentum of the Jupiter is $1.93x10^{43}kgm^{2}/s$, which is 70.96% of the total angular momentum for the solar system. The total angular momentum of the Saturn is $7.84x10^{42}kgm^{2}/s$, which is 28.89% of the total angular momentum for the solar system.

3. Collisions can be understood with simple considerations of the conservation of momentum, and energy. Two cases are to be illustrated, those involving *elastic* collisions (i.e. the kinetic energy of the colliding bodies is conserved) and *inelastic* collisions (i.e. some or all of the kinetic energy of the colliding bodies has been converted to other forms, e.g. heat). In either case, linear momentum is conserved until the system is acted on by an external source (in accordance with Newton's first law of motion). Thus by vector addition,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_3 + \mathrm{m}_2\mathbf{v}_4 \qquad \textit{elastic collisions}$$

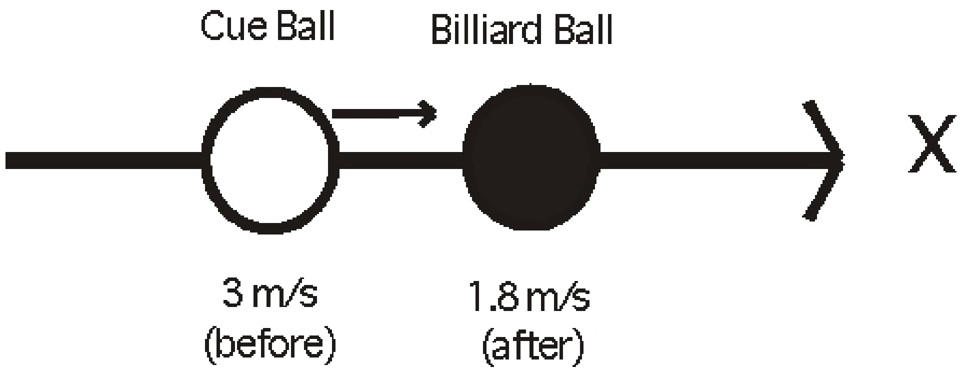
$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v}_{12} \qquad \textit{inelastic collisions}$$

where $m_1$ and $m_2$ are the mass of body one and two respectively, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the pre-impact velocities of body one and two, $\mathbf{v}_3$ and $\mathbf{v}_4$ are the post-impact velocities of body one and two, and $\mathbf{v}_{12}$ is the post-impact velocity of the combined body formed by the inelastic impact of bodies one and two.

We will use billiard balls to illustrate the elastic collisions (e.g. they deform negligibly under collision), planets, however, do not behave elastically but may be closely approximated as undergoing totally inelastic collisions (e.g. the two colliding bodies become one).

a) *Elastic Collisions*

A cue ball traveling with a **speed** (the magnitude of the velocity) of 3 m/s collides with a stationary billiard ball and imparts a speed of 1.8 m/s to the billiard ball. If the billiard ball is sent forward in the same line as the incident cue ball, what will be the speed of the cue ball after the collision? (Hint: Assume both balls have the same mass and conserve the system's momentum).




As noted above, the equation for elastic collsions is:

$$M_1V_1 + M_2V_2 = M_1V_3 + M_2V_4$$

The subscript for $M_1$ and $M_2$ can be dropped because the mass is the same for the cue ball and billiard ball. The given velocities can be inserted into the equation.

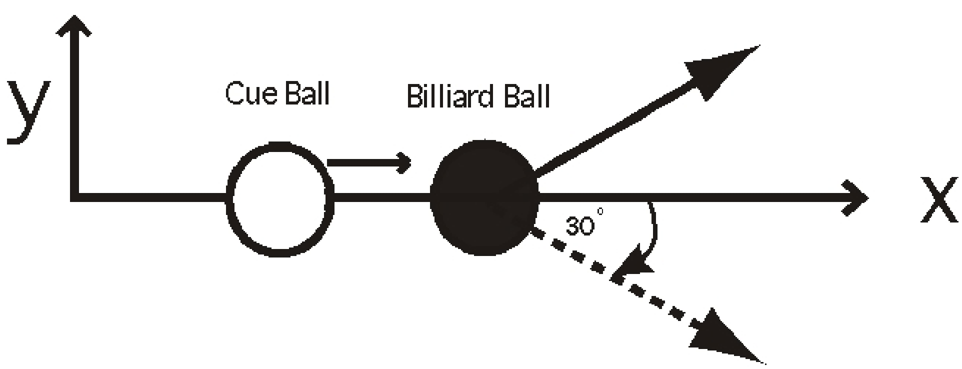
$$M(3m/s) + M(0m/s) = MV_3 + M(1.8m/s)$$

The masses cancel out. The equation can be rearranged to find $V_3$.

$$V_3 = 3m/s - 1.8m/s = 1.2m/s$$

b) *Velocity and Elastic Collisions*

Now assume that instead of a straight shot, you wish to send the billiard ball 30° to the side. This requires that you consider the magnitude *and* direction of the motion (termed **velocity**). Given the same two balls, and the initial cue ball velocity of 3 m/s along the x-direction (as shown below), and a velocity for the billiard ball of 1.8 m/s 30° off the x-direction (as shown). Determine the velocity (magnitude *and* direction) of the cue ball after collision (Hint: Conserve momentum independently in both the x- and y-direction).




If the billiard ball travels at a $30^o$, the equation changes to:

$$M(3m/s) + M(0m/s) = MV_{3x} + M(1.8m/s\cos(30^o))$$

where $V_{3x}$ is the cue ball velocity in the x-axis direction after it hit the billiard ball. The equation is rearranged to find the velocity of the cue ball in the x-axis direction.

$$V_{3x} = 3m/s - 1.8m/s\cos(30^o) = 1.44m/s$$

The velocity of the cue ball in the y-axis direction is:

$$V_{3y} = 1.8 m/s \sin(30^o) = 0.9 m/s$$

The velocity of cue ball is:

$$V_3 = \left( V_{3x}^2 + V_{3x}^2 \right)^{1/2} = 1.698 m/s$$

The direction of the cue ball is:

$$\theta = \tan^{-1}(0.9/1.44) = 32^o$$

c) *Inelastic Collisions*

Consider the collision of two bodies of Mass $M_1$ and $M_2$ that stick together upon collision. Let $M_2$ be at rest initially (as if you were standing on the second body watching the first), and let $\mathbf{v}_1$ be the velocity of $M_1$ before the collision.

i) Describe the motion of the system after the collision (give the magnitude and direction of the velocity after the collision).

$$M_1 V_1 + M_2 V_2 = (M_1 + M_2) V_3$$

$$M_1 V_1 + M_2 (0 m/s) = (M_1 + M_2) V_3$$

$$V_3 = \frac{M_1 V_1}{(M_1 + M_2)}$$

The direction is in the same direction as the initial velocity.

ii) What is the ratio of the final kinetic energy to the initial kinetic energy (Hint: Use the results from Part (i), you should end up with a function which is only dependent on the masses involved).

The Kinetic Energy (KE) is:

$$KE = \frac{1}{2} M V^2$$

$$KE_1 + KE_2 = KE_3$$

where $KE_1$ is the initial kinetic energy of the cue ball, $KE_2$ is the initial kinetic energy of the billard ball, and $KE_3$ is the final kinetic energy. $KE_2$ is zero because the inital velocity of the billard ball is zero. The above equation can be rearranged to find the ratio of inital and final KE.

$$\frac{KE_{final}}{KE_{inital}} = \frac{\frac{1}{2} M_1 V_1^2}{\frac{1}{2} M_3 V_3^2}$$

The mass in $M_3$ is $M_1 + M_2$. The variables can be cancelled out and $V_3$ from part i can be inserted into the above equation.

$$\frac{KE_{final}}{KE_{inital}} = \frac{(M_1 + M_2) \left( \frac{M_1 V_1}{M_1 + M_2} \right)^2}{\frac{1}{2} M_1 V_1^2} = \frac{M_1}{M_1 + M_2}$$

iii) What is the numerical value of this ratio for a planetesimal accreting into a planet (or for that matter for a meteorite striking the Earth)? Explain why this is and give a plausible explanation as to what has happened to it.

Assuming that the meteorite is $M_1$ and the earth is $M_2$, the ratio of final Kinetic energy and intial kinetic energy would be amost zero because the earth is much larger than the meteorite.

4. To learn about the The Near Earth Asteroid Rendezvous mission, go to the Geology 202 web page *http://www.earth.northwestern.edu/people/seth/202* and to the link for the Near Earth Asteroid Rendezvous (NEAR) mission under the topic *Meteorites, formation of the planets.* Read the information about this mission, and then the material about the asteroid Eros, NEAR's primary target. Explain briefly what NEAR found about Eros.

Next, go back to the course page, and to the link *Information on the NEAR Mathilde Flyby* and read the material, starting with the press release (you can also watch an animation of the Mathilde Flyby on the mainpage). Based on this, explain the goals of the NEAR Mathilde fly-by. What are some of the initial results?