GEOPHYSICAL RESEARCH LETTERS, VOL. 22, NO. 7, PAGES 743-746, APRIL 1, 1995

Kinematic considerations for mantle mixing

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Abstract. Recent experimental and computational studies show that "islands" (unmixed regions that do not interact with surrounding regions) are ubiquitous features in chaotically advecting fluids. Such islands quite naturally account for the geochemically inferred coexistence of apparently distinct, long-lived geochemical heterogeneity with relatively homogeneous regions of an actively convecting mantle. These results also indicate that mixing patterns—the set of islands and folds characterizing the large-scale material advection—are sensitive to small variations in the rheology of the fluid. Therefore, interpretation of numerical simulations of mantle transport and mixing is less straightforward than currently supposed. Computational studies of analytic flow solutions with systematically introduced and controlled errors indicate that mantle simulations are unlikely to accurately compute individual trajectories for even moderate time, but that trajectory ensembles can be accurately computed for long time. Significantly, computations also indicate that mixing and transport results may not evolve smoothly with increased rheological realism.

Introduction

Xenoliths, erupted basalts, and tectonically exposed ultramafic rocks provide evidence for mantle isotopic heterogeneity on scales from tens of meters to thousands of kilometers. Solid-state diffusion operating over the age of the earth can only homogenize $\mathcal{O}(10^{-2}\mathrm{m})$ scale heterogeneities. While mid-ocean ridge basalts exhibit small-amplitude heterogeneity on all scales, their relative global uniformity suggests they arise from a well-mixed reservoir. Oceanic island basalts (OIB), on the other hand, exhibit large-amplitude heterogeneity on large scales (e.g., the 1000-km scale DUPAL anomaly). In addition to a source region for mid-ocean ridge basalts (MORB), at least three additional reservoirs of distinct isotopic compositions, apparently associated in part with recycled crustal material, are required to explain the observed OIB mixing trends [*Hart and Zindler*, 1989].

Interpretation of these geochemical observations depends crucially on understanding the material advection (mixing and transport) properties of the active mantle. Experimental and computational mixing studies in highly controlled flows show that even seemingly inconsequential errors in the computed velocity field of 3-D or time-dependent flows can cause enormous changes in the advection patterns. These velocity field errors can arise from numerical sources and, more importantly, from imperfect rheological models. The results obtained to date are based on idealized flows, but their very simplicity makes them amenable to both in-depth theoretical study and a considerable degree of experimental investigation [*Ottino et al.*, 1988, 1992; *Ottino*, 1989, 1990a, 1990b; *Jana et al.*, 1994].

Computations

The fact that a flow is chaotic greatly accelerates the loss of accuracy, affecting not only the useful length of simulations, but also what quantities can even be meaningfully computed. Extensive computations point out the magnitude of these effects. There are two types of errors to be considered. The first are numerical errors, namely round-off, discretization, and integration errors. The second are model mismatch errors, such as specification of a rheology which does not exactly match that of the physical system. Both error sources are unavoidable in mantle computations. Flows with analytic solutions allow systematic, detailed investigation of the effects of all sources of error.

The most stringent computational test corresponds to an individual particle trajectory. After only 2 or 3 circulation times, an error in the velocity field greater than $\mathcal{O}(0.1\%)$ produces a deviation of the computed from the true trajectory which is of the order of the size of the domain. If, on the other hand we calculate the advection pattern (large folds, islands, striation patterns, etc.) by starting many points from nearly the same initial location, mimicking a dye advection experiment, velocity field errors of $\mathcal{O}(1\%)$ produce correct advection patterns: patterns identical to those obtained using analytical velocities and nearly indistinguishable from corresponding experiments. This can be understood by noting that errors in particle positions are *along* manifolds rather than normal to them, so that numerical trajectories continue to mark the manifolds which are therefore the robust features of chaotic flows. This behavior is a consequence of the sensitivity to initial conditions as expressed by the "shadowing lemma" [*Grebogi et al.*, 1990]. Simply, this lemma states that associated with almost every trajectory in the error-free dynamical system, there is a nearby trajectory in the simulated system that remains close to ("shadows") the actual trajectory for some (accuracy-dependent) duration. Computing many trajectories at once thus produces an accurate picture of the system at large.

Larger velocity field changes, above the 2–3% level with respect to some reference case, are produced by, for example, rheological or thermal variations. This is of particular

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Paper number 95GL00056
0094-8534/95/95GL-00056$03.00

Figure 1. Small changes in rheology cause substantial changes in large scale advection patterns. The flow is a 2-D Stokes flow in a box [*Leong and Ottino*, 1989]. The sidewalls are stationary while the top wall moves left to right and the bottom wall moves right to left. The walls move alternately to create time-dependent chaotically advecting flow. Results after 7 overturn periods for an initially compact dye blob [*Niederkorn and Ottino*, 1993; *Leong and Ottino*, 1990]. (a) Newtonian fluid (glycerin). (b) Slightly viscoelastic fluid (125 ppm polyacrylamide in glycerin). Large islands appear in the viscoelastic fluid where the Newtonian fluid is well-mixed.

relevance to mantle simulations for which the exact rheology is unknown. With a simulation the typical (often implicit) assumption is that if a rheological model is "close" in some sense to representing the mantle then the simulation results will be "close" as well. This assumption holds true for velocity profiles or streamlines, but it is definitely not true for the transport and mixing characteristics. This sensitivity of the transport to small changes in the velocity field is intrinsic to the nonlinear advection. To the extent that fully detailed simulations of the mantle are prohibitive and that parameters are only imperfectly known, it should be kept in mind, as can be inferred from Figure 1, that mixing results may not evolve smoothly with increased computational realism.

However, this does not imply that nothing can be reliably known. Although it is generally impossible to predict advection patterns in time-dependent flows by an examination of streamlines, mixing studies also show that advection patterns are built from only a few general features. Mixing in two-dimensional flows typically produces a combination of regular and chaotic regions. Although examples using time-periodic flows are more numerous, this general structure is also present in aperiodic and even turbulent flows [*Elhmaidi et al.*, 1993; *Babiano et al.*, 1994]. Regular regions—islands and chains of islands—stretch, rotate, and contract, but never mix with the surrounding chaotically advected fluid. Three-dimensional flows can be chaotic even with steady velocity fields, with islands corresponding to tubes of unmixed fluid shooting through the chaotic region (Figure 2).

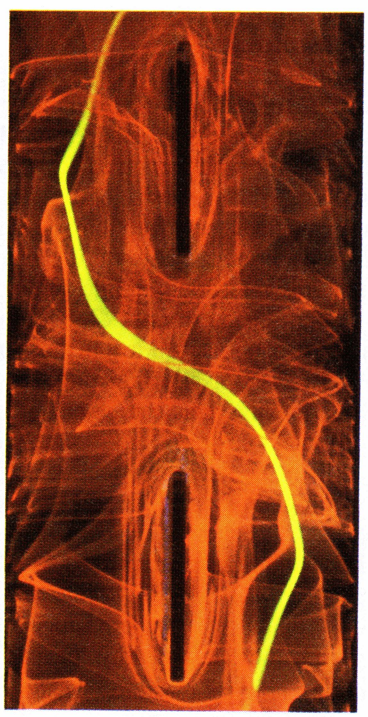

Figure 2. A stable flow tube surrounded by chaos in a section of steady 3-D laboratory flow [*Kusch and Ottino*, 1992]. Plates arranged periodically in the flow split and reorient the fluid while a circulation is imposed. Material in the tube does not mix while material in the chaotic region is well-mixed.

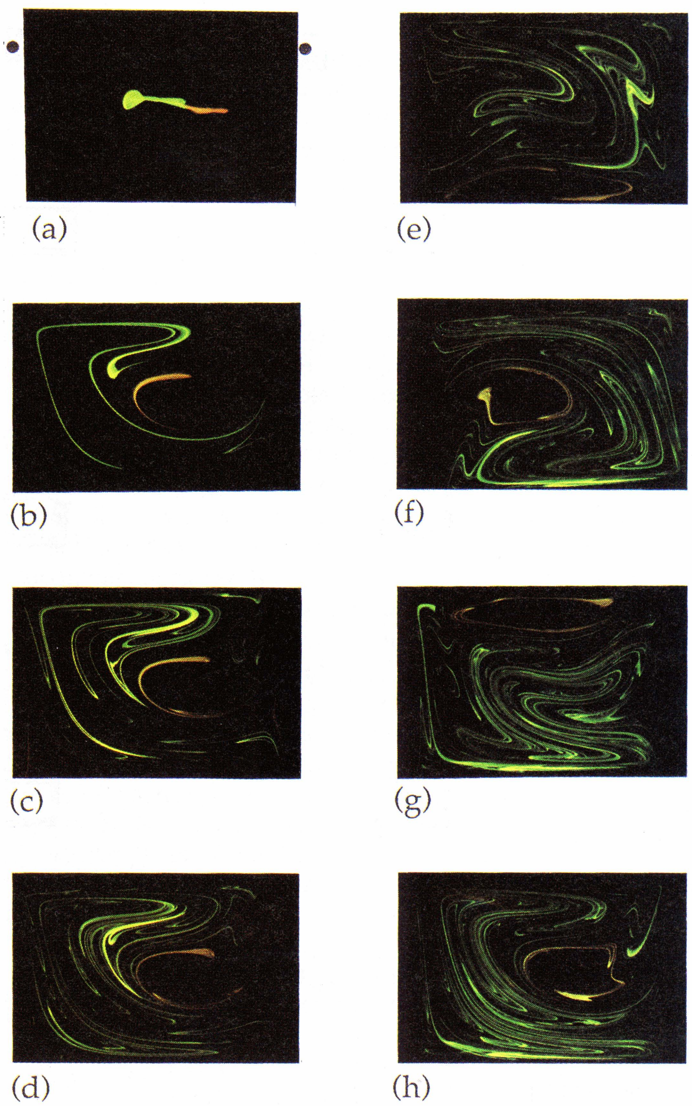

Figure 3. Chaotic advection in a box of Newtonian fluid with circulating streamlines similar to Figure 1. (a) Initial condition. (b-d) State after 2, 4 and 6 overturn times. (e-h) State after 6.25, 6.5, 6.75, and 7 overturn times.

Since these ideas are purely kinematic in nature, the particular flow is largely irrelevant. Furthermore, the qualitative advection results are independent of how the flow is forced: time-dependence can be induced just as well by boundary motions as by buoyancy or concentration fluxes. In fact *any* flux that induces a flow and might itself be time-varying [*Kellogg and Stewart*, 1991; *Gurnis*, 1986] or of such strength as to trigger a time-dependent instability produces the same general picture of regular and chaotic regions in motion. This happens with very simple, even analytically known, velocity fields, yet even so it is impossible to predict or understand the advection patterns by an examination of streamlines without the techniques developed in chaos/mixing theory.

Experiments

These general kinematic considerations can account quite naturally for certain geochemical observations, such as the persistence of distinct, long-lived geochemical reservoirs at a variety of scales in an actively convecting mantle. Models of layered, convectively isolated reservoirs often have been invoked [*Wasserburg and DePaolo*, 1979; *Anderson*, 1984] to explain mixing between layers via ascent of plumes from the transition zone [*Irifune and Ringwood*, 1993; *Ringwood*, 1994] or core-mantle boundary [*Hart et al.*, 1992] and/or by some degree of slab penetration into the lower mantle [*Silver et al.*, 1988; *Peltier and Solheim*, 1992]. However, the geophysical evidence for such compositional layering remains ambiguous [*Bina and Silver*, 1990]. Other models introduce heterogeneity by subduction, delamination, melting, entrainment, etc., allowing the reservoirs to exist as distributed parcels of material rather than as distinct layers [*Gurnis and Davies*, 1986; *Davies*, 1990]. *Kellogg* [1993] describes such processes in a recent review of mantle mixing.

Experiments (Figure 3) in a box of fluid show the coexistence of well-mixed and distinct heterogeneous source regions without invoking special hypotheses. Although laboratory experiments can only caricature mantle flow, plumes are the only qualitative features present in buoyancy driven flows which are missing from our laboratory flows. Figure 3a shows an initial line of the fluid dyed to visualize the subsequent mixing. Figures 3b-d show the mixing after the first three successive overturn times. The orange part of the line falls in an island, where it undergoes only weak rotation; the yellow-green part of the line is in the chaotic region, where it is exponentially stretched. The island appears stationary because pictures are taken at the end of each overturn. During an overturn period, the island moves throughout the domain (Figure 3e–h). The parameters of Figure 3 are in no way special: many other levels of energy input or flow-forcing methods, including non-periodic motions [*Elhmaidi et al.*, 1993; *Babiano et al.*, 1994], produce qualitatively the same picture of well-mixed regions and unmixed reservoirs in motion.

The structure of the dye composition along a fixed line may be measured through digital images similar to those of Figure 3. Consider an observation line fixed between the black dots in Figure 3a. Figure 4a shows the composition (grayscale light intensity from the dye) along this line as a function of overturn times. As time passes, the initial dye streak becomes stretched and folded by the flow so that the intersections with the test line become progressively thinner and more numerous. Figure 4b shows the corresponding Fourier transforms of compositional variation along the test line. The different slopes suggest different scaling regimes

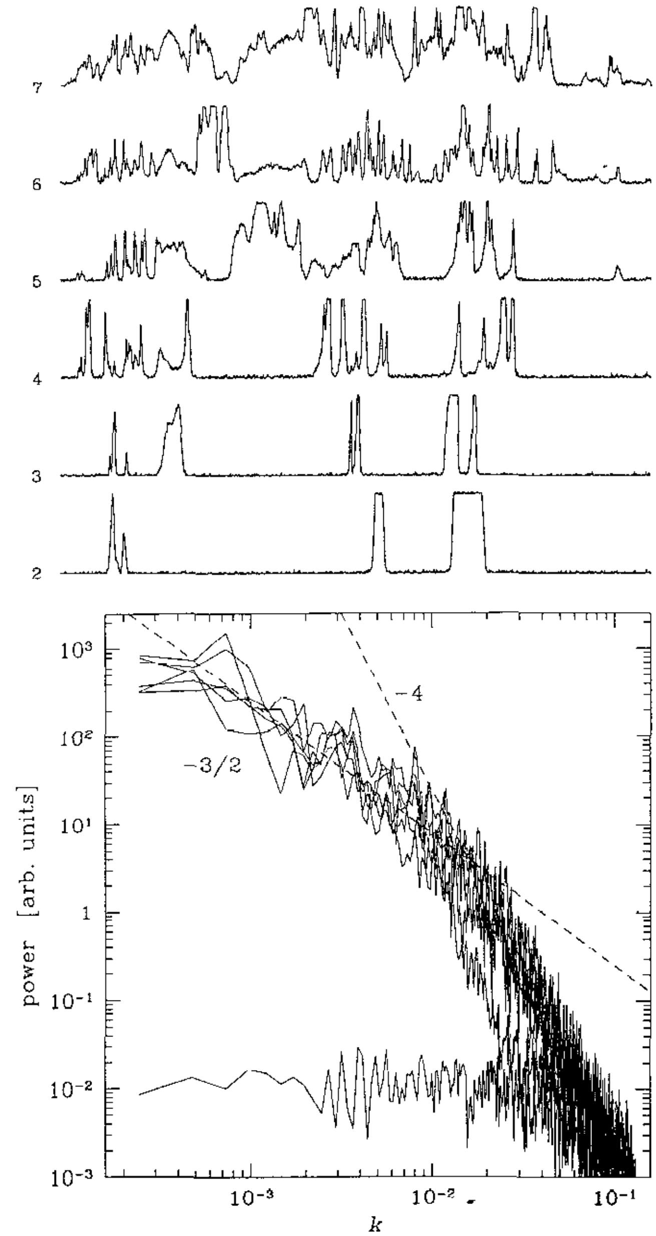

Figure 4. Even if field composition measurements were available in laboratory detail, one could still not reliably infer the underlying flow. (a) Dye composition along a line stretched across the box between the black dots in *a*. Overturn times are indicated on the left. The vertical scales in figure 4a are the same for each graph and indicate grayscale light intensity from the dye. As time passes, the initial dye streak becomes stretched and folded by the flow so that the intersections with a test line become progressively thinner and more numerous. Any number of different flows could produce the same refinement of striations. (b) Fourier spectra of (a). Lengths are scaled so that the box width is unity; the sampling frequency is 1/4096. The line at 10^{-2} power shows the measurement noise (i.e. the spectrum for Figure 3a). The spectra are quite similar for all times with two power-law regions. The dotted lines have slopes of -3/2 and 4.

for long- and short-range order. Unfortunately, even if field composition measurements as detailed as those possible in the laboratory were available, one still could not reliably infer the flow that produced the compositional structures: any number of different flows yield the same refinement of striations. What can be said, though, is that if the striation distribution is log-normal, the underlying flow is chaotically advecting [*Swanson et al.*, 1992].

Discussion

Several recent developments in mixing theory are relevant to questions arising in studies of mixing and transport in the mantle. Geochemical observations requiring long-term coexistence of isotopically distinct mantle source regions demand no special modeling requirements. Even a steady 3-D flow produces well-mixed and non-mixing regions that preserve long-lived distinct reservoirs, in addition to distributing the regions irregularly in less than ten circulation times. Computations of a chaotic flow requiring accurate individual trajectories are likely to be unreliable, but computations of large scale mixing structures (folds, islands, manifolds, etc.) based on the collective behavior of many trajectories are likely to be reliable. However, advection results do not respond smoothly to added realism in the simulation.

The ideas in this paper were developed using simple and rather restricted example flows. However, the results follow from rather general considerations of the kinematics of fluids and are, in particular, pertinent to mantle flows. Assessment of the ultimate impact of these results upon mantle studies must await their incorporation into future models and simulations.

Acknowledgments. We thank Bob Geller, Bill White, and an anonymous reviewer for helpful comments on the manuscript. C.R.B. acknowledges the support of National Science Foundation grant EAR-9158594.

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(received July 18, 1994; revised October 29, 1994; accepted November 29, 1994.)